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Connectivity and Planarity of Cayley Graphs

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Abstract. The question of which groups admit planar Cayley graphs goes back over 100 years, having been settled for finite groups by Maschke in 1896. Since that time, various authors have studied infinite planar Cayley graphs which satisfy additional special conditions. We consider the question of which groups possess any planar Cayley graphs at all by categorizing such graphs according to their connectivity.

Like planarity, connectivity is a fundamental concept in graph theory. We show that having a Cayley graph which is less than three–connected has strong implications for the structure of the group. In the planar case, the decomposition imposed by low connectivity allows us to reduce the problem to the case where the Cayley graph is three–connected, where geometric techniques can be employed.

1. Introduction.

Given a discrete group of isometries of the sphere, Euclidean plane or hyperbolic plane, one can construct a fundamental region whose translates tessellate the space. The dual of such a configuration is in fact a (modified) Cayley graph for the group. Figure 1 illustrates this construction for the group generated by the reflections in the sides of a spherical triangle with angles $\pi/2$, $\pi/3$ and $\pi/5$. Such a procedure gives rise to the familiar examples of Cayley graphs with a planar embedding. In 1896, Maschke [19] proved that the only finite

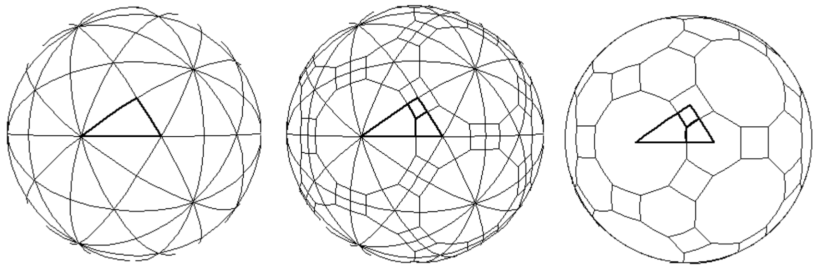

Figure 1: Constructing a planar Cayley graph.

groups which possess planar Cayley graphs are the discrete groups of isometries of the 2–sphere. In other words, every finite planar Cayley graph arises from a construction such as in Figure 1 for the sphere. Zieschang, Vogt and Coldewey [29] enumerated the infinite discrete groups whose Cayley complexes are planar. Of course, requiring that the Cayley complex be planar is more restrictive than requiring merely that the Cayley graph be planar. Several authors have considered groups which possess planar Cayley graphs subject to various other restrictions [14, 18, 27]. We shall be concerned with groups which possess any planar Cayley graph whatsoever, groups we will call *planar groups.*

We note that many authors have studied the *genus* of a finite group, that is, the minimum genus of any surface into which a Cayley graph for the group can be embedded, see [8]. Levinson [12] showed that if an infinite Cayley graph is not planar, then it cannot be embedded in any surface of finite genus.

Planarity is a Markov property, so by a classical theorem of Adyan and Rabin [1, 20] there is no algorithm which decides, given a presentation, whether the group so presented is planar (see, for example, [17, Chapter IV].) The best one can hope for is a "catalogue" of presentations, each of which presents a planar group, and such that each planar group is represented, and we show that the first step toward such a catalogue is to classify the Cayley graphs by their connectivity.

If a group possesses a planar Cayley graph which is three–connected, then the Cayley graph may be embedded in the sphere in such a way that the left action on the Cayley graph is realized as a group of homeomorphisms of the sphere, so the group may be analyzed by geometric techniques. On the other hand, we will show that a group with any Cayley graph of connectivity less than three has a natural action on a tree. If the Cayley graph is also planar, the component groups of the resulting graph of groups themselves have planar Cayley graphs of higher connectivity. This method is not only useful for studying planar Cayley graphs, but forms an effective program for studying actions of groups on planar graphs, see [22, 23].

2. Preliminaries.

Graphs. Recall that a *directed graph* consists of two sets, V (the set of *vertices*) and E (the set of *edges*), with two functions $\iota : E \to V$ and $\tau : E \to V$. The vertex $\iota(e)$ is called the *initial* vertex of e, and $\tau(e)$ is called its *terminal* vertex.

Let G be a group with generating set X (all groups in this paper will be assumed to be finitely generated.) The *(right) Cayley graph* $\mathcal{C}(G, X)$ is the directed graph with vertex set G and edge set $G \times X$, where the edge (g, x) has initial vertex g and terminal vertex gx. If $x \in X$ has order two, we identify each pair of edges (g, x) and (gx, x) into a single "unoriented edge." $\mathcal{C}(G, X)$ is sometimes referred to as a *modified* Cayley graph.

We shall be concerned only with the case where $\mathcal{C}(G, X)$ is *reduced*; that is, it has no loops and no multiple edges. This amounts to saying that $1 \notin X$, and if $x \in X$, then $x^{-1} \notin X$, unless x has order two.

Connectivity. A *separation* of a connected graph Γ is a collection of vertices whose removal separates Γ into two or more connected components. A separation consisting of n vertices will be called an *n–separation.* Γ is said to be *n–connected* if it has no $(n-1)$–separation; the largest n such that Γ is n–connected will be called the *connectivity* of Γ, denoted $\kappa(\Gamma)$.

Suppose Γ can be written as a (possibly infinite) union

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \cdots$$

of subgraphs Γ_i. If for each n, Γ_n has exactly one vertex in common with $\Gamma_1 \cup \cdots \cup \Gamma_{n-1}$, then we call Γ the *iterated one–point union* of the Γ_i, and if Γ_n has exactly one edge and no other vertices in common with $\Gamma_1 \cup \cdots \cup \Gamma_{n-1}$ for each n, we call Γ the *iterated one–edge union* of the Γ_i. We call either of these a *proper* decomposition if Γ_n is not a subgraph of $\Gamma_1 \cup \cdots \cup \Gamma_{n-1}$ for any n. It is well known that a graph is one–separable if and only if it has a proper decomposition as iterated one–point union. (On the other hand, a two–separable graph need not be a one–edge union since the vertices in a separation need not be adjacent.)

If G is a group, we define the *connectivity* of G to be the minimum connectivity of all Cayley graphs of G.

Planarity. A graph is called *planar* if and only if it can be represented by a collection of points and arcs in the two–sphere S^2 in such a way that two arcs do not intersect except possibly at their endpoints. If Γ is an embedding of some graph in S^2, then the connected components of $S^2 - \overline{\Gamma}$ are topological 2–disks called *regions.*

We shall require our embeddings to be tame in the following sense: first, if $\mathcal{A}$ is the set of accumulation points of *vertices* of Γ, then $\Gamma \cap \mathcal{A} = \emptyset$ and $\overline{\Gamma} = \Gamma \cup \mathcal{A}$. Second, we will require that the boundary of any region consist of a collection of disjoint paths of Γ, possibly together with some points of $\mathcal{A}$.

Following Levinson [15], we say the elements of $\mathcal{A}$ are *essential* if the boundary of any region contains at most one of them. In this case, the boundary of any region is either a circuit of Γ or it is a two–way infinite path of Γ together with the common accumulation point of its two ends. A region whose boundary is a (finite) circuit of Γ will be called a *finite* region; the others will be called *infinite* regions. For example, Figure 2 shows the Cayley graph of the group $Z \times Z_4 = \langle a, b \mid a^4 = 1, ab = ba \rangle$ embedded in the plane with two essential accumulation points. An infinite Cayley graph has one, two or infinitely many essential accumulation points [15]. Note that distinct ends of the group may correspond to the same essential accumulation point, as in the case of the free group of rank two whose Cayley graph (with respect to a free generating set) can be embedded in the sphere with a single accumulation point. On the other hand, the Cayley graph of the group $Z_2^3 *_{Z_2} Z_2^3$ with the obvious generating set is an infinite one–edge union of cubes (see Figure 3) which

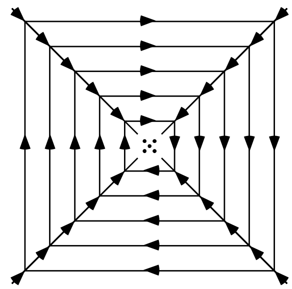

Figure 2: Part of a Cayley graph of $Z \times Z_4$

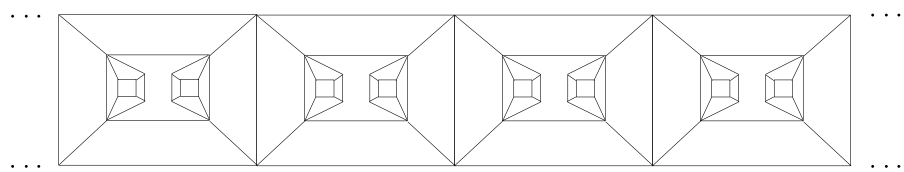

Figure 3: Part of a Cayley graph of the group $Z_2^3 *_{Z_2} Z_2^3$.

is hence planar, yet cannot be embedded with any finite number of accumulation points [6].

In general, if a graph Γ has a decomposition

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \cdots$$

as either iterated one–point union or iterated one–edge union, then Γ is planar if and only if each Γ_i is.

Groups acting on graphs. A *graph of groups* is a pair $(\mathcal{X}, \mathcal{G})$ where $\mathcal{X}$ is a directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, and $\mathcal{G}$ is a function which assigns to each $x \in \mathcal{V} \cup \mathcal{E}$ a group $\mathcal{G}(x)$, and two homomorphisms $\iota_e : \mathcal{G}(e) \to \mathcal{G}(\iota(e))$ and $\tau_e : \mathcal{G}(e) \to \mathcal{G}(\tau(e))$ for each $e \in \mathcal{E}$. If $x \in \mathcal{V}$, $\mathcal{G}(x)$ will be called a *vertex group,* and if $e \in \mathcal{E}$, it will be called an *edge group.*

For each $v \in \mathcal{V}$, let $\langle X_v \mid R_v \rangle$ be a presentation of $\mathcal{G}(v)$, where the X_v are taken to be pairwise disjoint. Then given a spanning tree $\mathcal{T} \subseteq \mathcal{X}$, we define the *fundamental group* $\pi_1(\mathcal{X}, \mathcal{T})$ to have the following presentation:

Generators:
- $\cup_{v \in \mathcal{V}} X_v$
- $\{t_e \mid e \in \mathcal{E}\}$, where the t_e are new symbols.

Relations:
- $\cup_{v \in \mathcal{V}} R_v$
- For each $e \in \mathcal{E}$ and for each $g \in \mathcal{G}(e)$, the relation
  $t_e^{-1} \iota_e(g) t_e = \tau_e(g)$.
- For each edge e of the spanning tree $\mathcal{T}$, the relation
  $t_e = 1$.

The generating set described above will be called the *standard* generating set for $\pi_1(\mathcal{X}, \mathcal{T})$.

We recall the following facts about this construction (see [4, 5, 21]): first, if $\mathcal{T}'$ is another spanning tree for $\mathcal{X}$, then $\pi_1(\mathcal{X}, \mathcal{T}') \simeq \pi_1(\mathcal{X}, \mathcal{T})$. Thus, we write $\pi_1(\mathcal{X})$.

Next, if all the homomorphisms ι_e and τ_e are injective, then so are the natural homomorphisms $\mathcal{G}(v) \to \pi_1(\mathcal{X})$ for each $v \in \mathcal{V}$. In this case, the graph of groups $(\mathcal{X}, \mathcal{G})$ is called *faithful*, and we may regard the vertex groups $\mathcal{G}(v)$ as subgroups of $\pi_1(\mathcal{X})$.

Let T be the directed graph whose vertices are the left cosets of the subgroups $\mathcal{G}(v)$ $(v \in \mathcal{V})$ in $\pi_1(\mathcal{X})$, and whose edges are the left cosets of the subgroups $\mathcal{G}(e)$ $(e \in \mathcal{E})$. The initial vertex of an edge $g\mathcal{G}(e)$ is the vertex $g\mathcal{G}(\iota(e))$ and its terminal vertex is $gt_e\mathcal{G}(\tau(e))$. T is a tree which is usually called the *standard tree* of the graph of groups $(\mathcal{X}, \mathcal{G})$.

Finally, let G be a group, and suppose that G acts on the left on a directed tree T. Let $\mathcal{X} = G \backslash T$. Choose a spanning tree $\mathcal{T}$ in $\mathcal{X}$ and a connected preimage $\tilde{\mathcal{T}}$ of $\mathcal{T}$ in T; for any vertex or edge x of $\mathcal{T}$, denote its preimage in $\tilde{\mathcal{T}}$ by $\tilde{x}$. Also, for each edge $e \in \mathcal{X} - \mathcal{T}$, choose an edge $\tilde{e} \in T$ whose initial vertex lies in $\tilde{\mathcal{T}}$.

For each vertex or edge x of $\mathcal{X}$ let $\mathcal{G}(x)$ be the stabilizer of $\tilde{x}$ under the G–action on T. If $e \in \mathcal{T}$, then we may take the homomorphisms ι_e and τ_e to be the inclusions, since an element of G stabilizes an edge of T only if it stabilizes both its endpoints. If $e \in \Gamma - \mathcal{T}$, choose an element $g_e \in G$ so that $g_e\widetilde{\tau(e)} = \tau(\tilde{e})$. In this case, define ι_e to be inclusion, and τ_e to be conjugation by g_e. Then $\pi_1(\mathcal{X}, \mathcal{T})$ is isomorphic to G, and the standard tree associated to $\mathcal{X}$ is T.

Let $(\mathcal{X}, \mathcal{G})$ be a graph of groups with $G = \pi_1(\mathcal{X})$ and let X be the standard generating set for G. To each path P in the Cayley graph $\mathcal{C}(G, X)$ we associate a path $\tilde{P}$ in the standard tree T as follows: first, choose a vertex v_0 of $\mathcal{X}$. If P begins at g and ends at h in $\mathcal{C}$, $\tilde{P}$ will be the unique reduced path in T from $g\mathcal{G}(v_0)$ to $h\mathcal{G}(v_0)$.

Now, it is straightforward to verify that if $\tilde{P}$ traverses an edge $g\mathcal{G}(e)$ for some edge e of $\mathcal{X}$ and some $g \in G$, then some vertex on the path P must belong to the coset $g\mathcal{G}(e) \subseteq G$. (This is clear if P has length one, and if P_1 ends at the beginning point of P_2, then any edge traversed by $\widetilde{P_1.P_2}$ is traversed by at least one of $\tilde{P}_1$ or $\tilde{P}_2$.)

Since removing any edge from T separates T, we have:

Theorem 2.1 *If G is the fundamental group of a faithful graph of groups, then in the Cayley graph of G with respect to the standard generating set, the elements of any edge group form a separation.*

Planar groups. If a group G possesses a generating set X such that $\mathcal{C}(G, X)$ is planar, then we will say that G is planar, and that X is a planar generating set. We define the *planar connectivity* $\kappa_p(G)$ of a planar group G to be the minimum connectivity of all its planar Cayley graphs. Since a Cayley graph is connected by definition, $\kappa_p(G) \geq 1$ for every planar group G.

Theorem 2.2 (Babai [2]) *Any subgroup of a planar group is planar.*

In fact, suppose that Γ is a Cayley graph for G and that H is a subgroup of G. Then one can construct a Cayley graph for H as follows: choose a spanning tree T for the quotient $H \backslash \Gamma$ of the H–action on Γ, and contract each preimage of T in Γ to a point. Babai proves that H acts on the resulting graph Γ'. The action is free and transitive on the vertices of Γ', and so Γ' is a Cayley graph for H. Clearly Γ' is planar if Γ is.

In fact, the following is an immediate corollary of his proof:

Corollary 2.3 *If $\mathcal{C}(G,X)$ is planar and H is a subgroup of G, then H has a generating set Y with $H \cap X \subseteq Y$ such that $\mathcal{C}(H,Y)$ is planar.*

3. Groups of low connectivity.

In this section we investigate what can be concluded about the structure of a group assuming only that it possesses a Cayley graph of low connectivity.

One can describe the structure of a connected 1–separable graph Γ by forming its *block–cutpoint* tree $\mathcal{B}$, see [24]. $\mathcal{B}$ has a vertex for each cutpoint of Γ, and one for each *block*, that is, for each maximal two–connected subgraph, of Γ. A vertex corresponding to a block is adjacent to one corresponding to a cutpoint if and only if the block contains the cutpoint. If a group acts on Γ then the action extends naturally to an action on $\mathcal{B}$, and so the Bass–Serre theory can be used to study the structure of the group.

Theorem 3.1 *A group G possesses a 1–separable Cayley graph if and only if G is either infinite cyclic or a non–trivial free product of infinite cyclic groups and groups of connectivity two or more.*

Proof. Let $\mathcal{C}$ be a Cayley graph for G. If $\mathcal{C}$ has a cut vertex, then *every* vertex of $\mathcal{C}$ is a cutpoint. Let $\mathcal{B}$ be the block–cutpoint tree of $\mathcal{C}$. G acts on the left on $\mathcal{C}$, and therefore on $\mathcal{B}$. No element of G stabilizes any vertex of $\mathcal{C}$, so the G–stabilizer of any edge of $\mathcal{B}$ is also trivial, since an element of G can fix an edge of $\mathcal{B}$ if and only if it fixes both endpoints of the edge. G fixes no block of $\mathcal{C}$ either, since any such block would have to contain every vertex of $\mathcal{C}$, contradicting the 1–separability of $\mathcal{C}$.

Thus, G is the fundamental group of a graph of groups whose underlying graph is the quotient $G\backslash\mathcal{B}$, whose edge groups are all trivial and whose vertex groups are all proper subgroups of G. Therefore, G is either a non–trivial free product or it is infinite cyclic. Moreover, G can be written as the free product of freely indecomposable groups, and a freely indecomposable group is either infinite cyclic or has connectivity at least two.

The converse is an immediate consequence of Theorem 2.1. □

Example 3.2 Consider the group $Z_4 * Z_2 = \langle a, b \mid a^4 = b^2 = 1\rangle$, whose modified Cayley graph is depicted in Figure 4, in which we see that the free product structure exhibited by the action on the block–cutpoint tree is already apparent in the Cayley graph. Note that the blocks are themselves Cayley graphs. □

Example 3.3 The Cayley graph of $\langle a, b \mid abab = 1\rangle$ is also 1–separable, see Figure 5, and the action on the block–cutpoint tree makes the free product decomposition apparent: the quotient of the block–cutpoint tree by the group action is a graph with two vertices joined by two edges. The stabilizers of the (solid) block vertices have order 2 and all the other stabilizers are trivial, and so by the Bass–Serre theory the group is isomorphic to $Z * Z_2$. Note that the blocks are not themselves Cayley graphs in this case. □

Before discussing groups which possess two–separable Cayley graphs, we record some facts about graphs of connectivity two. The reader is referred to [24] for details about this in the case of finite graphs, and to [7] for infinite graphs.

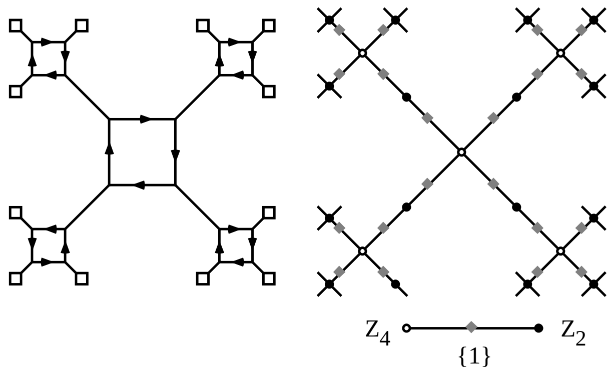

Figure 4: The Cayley graph of $Z_4 * Z_2$, its block–cutpoint tree and quotient.

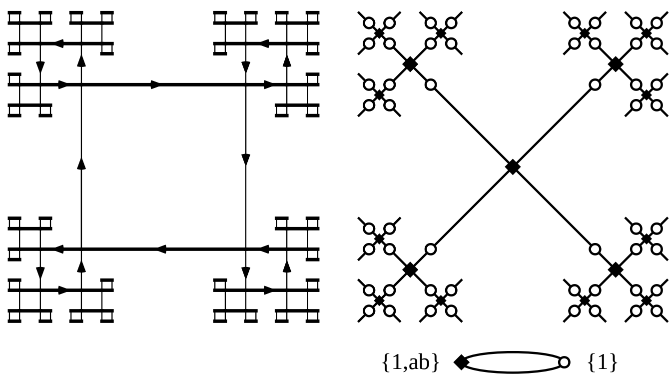

Figure 5: $\langle a, b \mid abab = 1\rangle$.

The block–cutpoint tree encodes how to construct a one–separable graph from two–connected pieces using the operation of vertex union. There is an analogous description for two–separable graphs. In this case the pieces are *three–blocks*, which may be either three–connected graphs, circuits, or *n–links*, that is, two vertices joined by $n \geq 3$ parallel edges. The operation analogous to vertex union is called *edge amalgamation,* under which two three–blocks are joined together along an edge—called a *virtual edge*—which is then deleted. Since the virtual edges are deleted, the three–blocks need not be subgraphs of Γ. They are, however, homeomorphic to subgraphs of Γ, so Γ is planar if any only if all its three–blocks are, and in fact the graph obtained from Γ by adjoining any virtual edges is also planar in this case. As in the case of the block–cutpoint tree, each element of $\mathrm{Aut}(\Gamma)$ induces an automorphism of the three–block tree.

Lemma 3.4 *Let G be a group which has a two–separable Cayley graph $\mathcal{C} = \mathcal{C}(G, X)$ with respect to some generating set X. Then $G = \pi_1(\mathcal{X})$ for some graph of groups $(\mathcal{X}, \mathcal{G})$, where $|\mathcal{G}(e)| \leq 2$ for each edge e of $\mathcal{X}$.*

Proof. Let $\mathcal{B}$ be the 3–block tree of $\mathcal{C}$. Since G acts on $\mathcal{C}$, it acts on $\mathcal{B}$. As before, G is the fundamental group of the graph of groups $(\mathcal{X}, \mathcal{G})$, where $\mathcal{X} = G\backslash\mathcal{B}$.

Now, since the stabilizer in G of any virtual edge of $\mathcal{C}$ has order one or two, and since each edge of $\mathcal{B}$ is incident to a vertex corresponding to a virtual edge of $\mathcal{C}$, it follows that the stabilizer in G of any edge of $\mathcal{B}$ has order at most two. □

Lemma 3.5 *If $\mathcal{C}$ has more than one three–block, then $(\mathcal{X}, \mathcal{G})$ is nontrivial; that is, G stabilizes no three–block of $\mathcal{C}$ and no virtual edge of $\mathcal{C}$.*

PROOF. G acts transitively on the vertices of $\mathcal{C}$, so if G stabilizes a virtual edge, then $|G| = 2$ and $\mathcal{C}$ can have only one three–block.

If G stabilizes a 3–block b of $\mathcal{C}$, then b must contain all the vertices of $\mathcal{C}$. Therefore, each 3–block other than b contains exactly two vertices, and is thus an n–link for some n. Since two n–links cannot be adjacent to one another, see [7], each 3–block other than b is adjacent only to b. Therefore, each three–block other than b is a link with one virtual edge and two or more "real" edges, which is impossible since $\mathcal{C}$ is reduced. □

A theorem of Linnell [16] says that if n is a positive integer, then any finitely–generated group has a maximal expression as the fundamental group of a graph of groups whose edge groups have order n or less; that is, one whose vertex groups cannot be so decomposed. This allows us to conclude:

Theorem 3.6 *If $\kappa(G) = 2$ then either G is a finite cyclic or dihedral group, or it is the fundamental group of a graph of groups whose edge groups have order two or less and whose vertex groups have connectivity at least three.*

PROOF. Let $\mathcal{C}$ be a Cayley graph for G with connectivity two. If $\mathcal{C}$ has only one three–block, then it is a finite polygon, and so G is either a finite cyclic or a finite dihedral group. Otherwise, it follows from Lemmas 3.4 and 3.5 that G has such a decomposition, and from Linnell's theorem we may conclude that it has one whose vertex groups have connectivity at least three. □

Example 3.7 Figure 6 shows the Cayley graph of the group

$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (bc)^2 = 1\rangle$$

in which the stabilizers of the hexagon 3–blocks are conjugate to $A = \langle a, b \mid a^2 = b^2 = (ab)^3 = 1\rangle$, the stabilizers of the quadrilateral 3–blocks are conjugate to $C = \langle b, c \mid b^2 = c^2 = (bc)^2 = 1\rangle$, and the stabilizers of the virtual edges are conjugate to $B = \langle b \mid b^2 = 1\rangle$, corresponding to the representation of the group as $A *_B C$. Note that the 3–blocks are all Cayley graphs. □

Example 3.8 The stabilizers of the virtual edges may also be trivial, as in the case of $\langle a, b \mid (ab^2)^2 = 1\rangle$, whose Cayley graph is drawn in Figure 7 in which the curved edges correspond to generator a and the straight edges correspond to generator b. Examining the 3–block tree reveals the isomorphism with the free product $Z * Z_2$. □

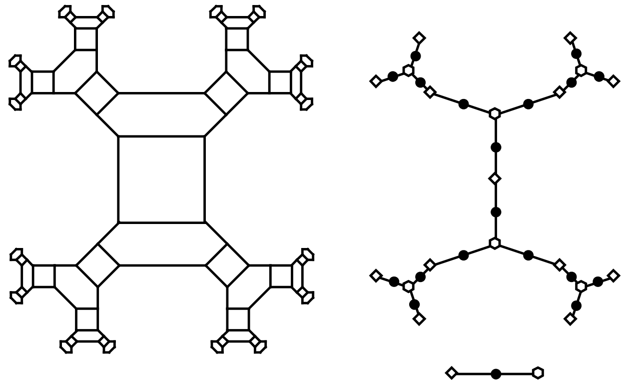

Figure 6: A 2–connected Cayley graph.

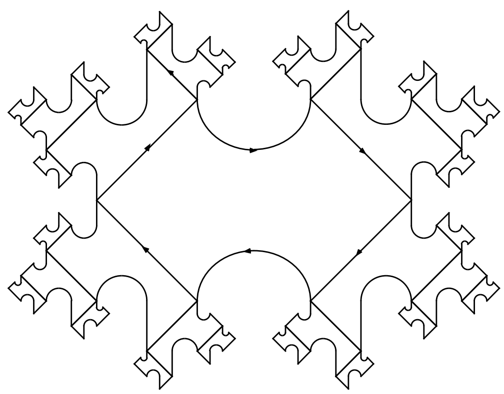

Figure 7: A 2–connected Cayley graph.

4. Planar connectivity.

4.1. Low Planar connectivity.

We turn again to planar Cayley graphs.

Theorem 4.1 *A group G has a 1–separable planar Cayley graph if and only if it is either infinite cyclic or a non–trivial free product of planar groups.*

Proof. Choose a generating set X for G such that $\mathcal{C}(G, X)$ is planar and one–separable. If G is not infinite cyclic, then it is a non–trivial free product $G_1 * G_2$, and each G_i is planar, by Babai's Theorem.

Conversely, suppose that G has such a decomposition. Choose a generating set X_i for each G_i so that the Cayley graphs $\mathcal{C}_i = \mathcal{C}(G_i, X_i)$ are both planar, and let $X = X_1 \cup X_2$. Then it is straightforward to verify that $\mathcal{C}(G, X)$ is the iterated one–point union of subgraphs isomorphic to the $\mathcal{C}_i$, and so it is planar and 1–separable. □

In fact, we can say a little more. Since any finitely generated group can be written as a free product of freely indecomposable groups, we have

Corollary 4.2 *If G is planar with $\kappa_p(G) = 1$, then G is either infinite cyclic, or it is a free product of planar groups of planar connectivity two or more.*

Finally, we record the following fact:

Proposition 4.3 *If G has one Cayley graph which is 1–separable, and another which is planar, then $\kappa_p(G) = 1$.*

Proof. This is clear if G is infinite cyclic. Otherwise, the first condition implies that G is a non–trivial free product, and the second condition and Babai's theorem imply that the free factors are planar. As in the proof of Theorem 4.1, G has a planar, one–separable Cayley graph. □

Theorem 4.4 *If G has planar connectivity 2, then either G is a finite cyclic or dihedral group, or it is the fundamental group of a graph of groups whose edge groups all have order two or less and whose vertex groups all have planar connectivity at least three. In the latter case, the vertex groups have planar generating sets which include the nontrivial elements of the incident edge groups.*

Proof. If G has a 2–separable Cayley graph with only one three–block, then it is a finite polygon, and so G is a finite cyclic or dihedral groups. Otherwise, Theorem 3.4 guarantees that G has the given structure. Corollary 2.3 guarantees that the vertex groups are planar, and the statement about the generating sets of the vertex groups follows from the fact that a two–separable planar graph remains planar when the virtual edges are adjoined. □

It is conceivable that a planar group may have connectivity two but planar connectivity greater than two. However, if $G = G_1 *_A G_2$, where A has order two, and if there are planar generating sets X_1 and X_2 for G_1 and G_2, respectively, each of which contains the nontrivial element of A, then it is easy to see that $\mathcal{C}(G, X_1 \cup X_2)$ is an iterated one–edge union of subgraphs isomorphic to $\mathcal{C}_1$ and $\mathcal{C}_2$, and so it has planar connectivity two.

In the case of an HNN extension $G_1 *_A$, if G_1 has a planar generating set X_1 containing both the generator of A and its conjugate by the stable letter t of the HNN extension, then the Cayley graph of G with respect to the generating set $X_1 \cup \{t\}$ is easily seen to be the iterated one–edge union of subgraphs isomorphic to $\mathcal{C}_1$ with quadrilaterals labeled $atbt^{-1}$, and so it is planar and has connectivity two in this case, as well.

4.2. Higher planar connectivity.

We begin with a theorem about three–connected planar graphs:

Theorem 4.5 *Let Γ be a three–connected planar graph, and let c be a simple circuit of Γ. If c bounds a disk in* some *planar embedding of Γ, then it does so in* every *such embedding.*

Proof. This is an immediate consequence of Whitney's theorem on the unique embeddability of planar three–connected graphs (see [26] for a proof in the finite case, and [11] for the infinite case.) □

Thus we may say unambiguously that a particular circuit of a three–connected planar Γ "bounds a disk."

Corollary 4.6 *If c bounds a disk, then so does its image under any automorphism of Γ. In particular, if Γ is a Cayley graph, and if w is the boundary label of a circuit which bounds a disk, then* every *circuit with label w bounds a disk.*

Let G be an infinite group with generating set X, let Γ be a tame embedding of the Cayley graph of G in S^2, and let $\mathcal{A}$ be the set of accumulation points of vertices of Γ. Suppose also that Γ is three–connected and that $\mathcal{A}$ is essential.

If $|\mathcal{A}| = 1$, then it is easy to see that the union of Γ with the disks bounded by circuits of Γ is in fact a (planar) Cayley *complex* for G, and so G is either a "planar discontinuous group" (studied by Zieschang, Vogt and Coldewey [29]) or a "non–Euclidean crystallographic group" studied by MacBeath [18] (see also [17, Chapter 3].) We will refer to such groups collectively as "plane groups."

If $|\mathcal{A}| = 2$, then Γ is a so–called "cylindric embedding" of the Cayley graph of G, which has been studied by Levinson. (Note that although there is an error in this paper [28], the results are correct if the Cayley graph is also assumed to be three–connected.) In particular, G is a one–relator quotient of a plane group.

If $|\mathcal{A}| > 2$, then $\mathcal{A}$ is in fact infinite [15]. Let $\mathcal{U}$ be the universal cover of $S^2 - \mathcal{A}$ (which exists since $\mathcal{A}$ is closed) and let $\tilde{\Gamma}$ be the preimage of Γ under the covering map $p : \mathcal{U} \to S^2 - \mathcal{A}$. Let $\mathcal{D}$ be the subset of S^2 consisting of Γ and all the finite regions. Then it is easy to see that $\mathcal{D}$ has the structure of a connected 2–complex whose 1–skeleton is Γ. Furthermore, the boundary of each infinite region consists of a two–way infinite path of Γ and one accumulation point, and so $S^2 - \mathcal{A}$ contracts onto $\mathcal{D}$.

Proposition 4.7 *$\tilde{\Gamma}$ is connected.*

Proof. Since $S^2 - \mathcal{A}$ contracts onto $\mathcal{D}$, any path in $S^2 - \mathcal{A}$ whose endpoints lie in Γ is homotopic to one lying in Γ. Therefore, the same is true of any path in $\mathcal{U}$ joining two points of $\tilde{\Gamma}$; in particular, $\tilde{\Gamma}$ is connected. □

Let S be the set of labels of circuits in Γ which bound (finite) regions, and let $\tilde{G}$ be the group with presentation $\langle X \mid S \rangle$.

Theorem 4.8 *$\tilde{\Gamma}$ is the Cayley graph of $\tilde{G}$ relative to the given generating set.*

Proof. It is clear that any path in $\tilde{\Gamma}$ whose label belongs to S is a loop, and so it will suffice to prove that the label of *any* loop in $\tilde{\Gamma}$ can be written as a product of conjugates of elements of S. Let ℓ be such a loop. Then ℓ is contractible in $\mathcal{U}$. But this implies that its image $p(\ell)$—which has the same label as ℓ—is contractible in $S^2 - \mathcal{A}$. But since $S^2 - \mathcal{A}$ contracts onto $\mathcal{D}$, $p(\ell)$ is contractible in $\mathcal{D}$, which implies that its label is a product of conjugates of region boundaries. □

Now let N be the group of covering transformations for the covering $p : \mathcal{U} \to S^2 - \mathcal{A}$.

Theorem 4.9 *N is isomorphic to $K = \ker(\tilde{G} \to G)$.*

PROOF. Let $\alpha \in N$, and let ℓ be the label of the path in $\tilde{\Gamma}$ from the vertex "1" to $\alpha(1)$. Since $p(1) = p(\alpha(1))$ in $\Gamma \subseteq S^2 - \mathcal{A}$, it is clear that the word ℓ represents an element of $\tilde{G}$ which maps to $1 \in G$.

Conversely, if W is a word representing an element of K, then the path in Γ with label W and beginning at 1 is a loop, ℓ. But then there is a unique covering transformation mapping $1 \in \tilde{\Gamma}$ to the other endpoint of the preimage $\tilde{\ell}$ of ℓ in $\mathcal{U}$.

Finally, it is easy to check that these correspondences are indeed homomorphisms. □

Now, one may identify an infinite plane group with a discrete group of motions of either the Euclidean plane R^2 or the hyperbolic plane H^2, and so N would be a discrete group acting on the same space and having a planar quotient. Thus, one approach to the problem of producing a catalogue of planar groups would be to try to catalogue the groups of motions of R^2 and H^2 which have planar quotient spaces. (Along these lines, one knows a condition on such a group which guarantees that its quotient is orientable [10].)

5. Geometrization

Given a planar Cayley graph, is there always a nice picture for it? In other words, is there always an equivariant embedding of the Cayley graph in a convenient geometric surface: the sphere, Euclidean plane, hyperbolic plane, infinite cylinder, or infinitely punctured hyperbolic plane, so that the action of the group on the Cayley graph is realized by isometries of the space? We know that this always works for 3–connected planar Cayley graphs, as well as for 1–separable Cayley graphs all of whose blocks are 3–connected. For 2–connected planar Cayley graphs, however, there need not exist an equivariant embedding.

Theorem 5.1 *There exists Cayley graphs which are planar and* 2*–connected, but which have no embedding in the sphere for which the natural left action on the Cayley graph is realized by homeomorphisms.*

In Figure 8 we see the construction of a planar Cayley graph for the free product with

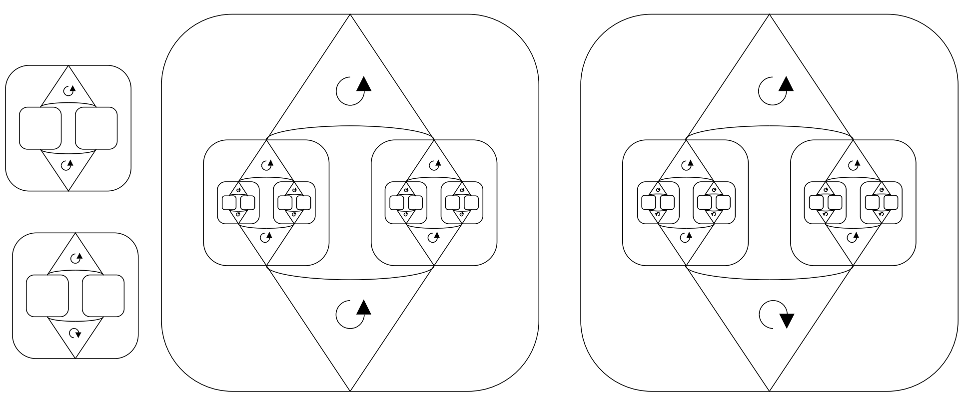

Figure 8:

Z_2–amalgamation of D_3 and $Z_3 \times Z_2$. The Cayley graphs of both factors are triangular prisms, pictured on the left in Figure 8 with the orientations of the edges of the triangles indicated, and the spokes of the prism, representing involutory edges, indicated by 2–gonal cells. The Cayley graph of the amalgamation embeds in the sphere, with the upper and lower hemispheres shown. Since the Z_2 action in $Z_3 \times Z_2$ on the prism is by reflection, while the Z_2 action in D_3 on the prism is by rotation, amalgamating these incompatible actions yields a planar Cayley graph which has no equivariant embedding.

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