Cylindric Embeddings of Cayley Graphs

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Abstract

An embedding of an infinite Cayley graph in the two–sphere has either one, two, or an infinite number of essential accumulation points of vertices. We obtain a list of group presentations which includes every group possessing a Cayley graph that can be embedded in the two–sphere with two essential accumulation points of vertices.

Let G be a group and let X be a generating set for G. Then the (right) *Cayley graph* $\mathscr{C}(G,X)$ is defined to be the directed graph with vertex set G and edge set $G \times X$, where the edge (g,x) points from vertex g to vertex gx. If $x \in X$ has order two, then we replace the oppositely directed edges (g,x) and (gx,x) by a single undirected edge joining g and gx. In this case, we refer to $\mathscr{C}(G,X)$ as a *modified* Cayley graph. We shall be interested in finitely–generated groups which possess planar Cayley graphs. It is clear that a modified Cayley graph is planar if and only if its unmodified counterpart is.

The finite groups with planar Cayley graphs were determined by Maschke [7]; they are just the finite groups of isometries of the 2-sphere, including those that contain reflections.

If G is infinite and $\mathscr{C} = \mathscr{C}(G,X)$ is planar, then any embedding of $\mathscr{C}$ in the 2-sphere will have accumulation points of vertices. Following H. Levinson [5], we will say these accumulation points are *essential* if given any two of them, there is a circuit of $\mathscr{C}$ that separates them; that is, there is a circuit in $\mathscr{C}$ which corresponds to a Jordan curve **C** in the embedding, and the two accumulation points lie on opposite sides of **C**. Levinson proved in [5] that the number of essential accumulation points is either 1 or 2, or it is infinite. (In this way, the essential accumulation points are somewhat likes ends of G, though the number of ends may be strictly greater than the number of essential accumulation points. For example, a non-cyclic free group has infinitely many ends, but its Cayley graph with respect to a free basis, being a tree, can be embedded in the sphere with a single accumulation point of vertices.)

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Our goal is to determine the finitely–generated groups which possess Cayley graphs that can be embedded in the sphere with 2 essential accumulation points of vertices; that is, they can be embedded without accumulation points of vertices in an infinite cylinder. We will restrict our attention to Cayley graphs that are 3-connected, for if G has a Cayley graph of connectivity 1 or 2, then it can be decomposed as either a free-product-with-amalgamations $A *_H B$ or an HNN-extension $A*_H$, where where A and B have planar Cayley graphs and H has order 1 or 2 [2]. (We remind the reader that a graph is said to be "n–connected" for some integer n if deleting any subset of n or fewer vertices, along with all the edges incident to them, leaves a connected graph. A graph is said to have "connectivity n" if it is n–connected but not $(n+1)$–connected.)

So suppose $\mathscr{C} = \mathscr{C}(G,X)$ is 3–connected, and that it is embedded in the sphere with two essential accumulation points of vertices, call them "**n**" and "**s**." Then each vertex of $\mathscr{C}$ has degree ≥ 3, and there is a circuit ℓ in $\mathscr{C}$ which separates **n** and **s** in the embedding.

We assume further that the embedding is tame in the following sense: first, the closure $\overline{\mathscr{C}}$ of $\mathscr{C}$ is the union $\mathscr{C} \cup \{\mathbf{n},\mathbf{s}\}$; second, the complement of $\overline{\mathscr{C}}$ in $\mathbb{S}^2$ consists of pairwise disjoint subsets of $\mathbb{S}^2$—called "regions"—which are homeomorphic to open 2–disks; and finally, the boundary of any region is homeomorphic to a circle.

Let us call a region *finite* if its boundary is a finite circuit of $\mathscr{C}$, and *infinite* otherwise. Since the accumulation points **n** and **s** are essential, the boundary of an infinite region contains exactly one of them, and is therefore the union of a two–way infinite path in $\mathscr{C}$ with the common accumulation point of its two ends.

Theorem 1 *There are no infinite regions.*

PROOF. We refer the reader to Figure 1. Suppose there is an infinite region. Then since $\mathscr{C}$ is 3–connected, every vertex of $\mathscr{C}$ must lie on the boundary of an infinite region [4]. Let $\mathscr{N}$ be the hemisphere of $\mathbb{S}^2 \backslash \ell$ which contains **n**.

Let v be a vertex of $\mathscr{C}$ which lies in $\mathscr{N}$, and whose distance to ℓ is ≥ 2, and let $\mathscr{R}$ be an infinite region whose boundary contains v. Then $\mathscr{R} \subseteq \mathscr{N}$, and the boundary of $\mathscr{R}$ is the union of **n** and a two-way infinite path of $\mathscr{C}$ whose two ends both approach **n**.

Since v has degree ≥ 3, we may choose a vertex u which is adjacent to v but which does not lie on the boundary of $\mathscr{R}$. Then $u \in \mathscr{N}$ and u lies on the boundary of a different infinite region $\mathscr{R}'$. Once again, $\mathscr{R}' \subseteq \mathscr{N}$ and the boundary of $\mathscr{R}'$ is the union of **n** and another two-way infinite path whose two ends both approach **n**. Let A be an arc in $\mathscr{R}$ joining v to **n** and let A' be an arc in $\mathscr{R}'$ joining u to **n**. Then it is clear that the union of the arcs A and A', the point **n**, and the edge (v,u) is a Jordan curve which separates the sphere into two components, each of them containing vertices of $\mathscr{C}$. That is, $\{v,u\}$ is a separation of $\mathscr{C}$, contradicting the assumption that $\mathscr{C}$ is 3–connected. □

Let $\mathscr{B}$ be the set of labels of circuits in $\mathscr{C}$ that are boundaries of regions. Since $\mathscr{C}$ is 3–connected, every circuit in $\mathscr{C}$ whose label lies in $\mathscr{B}$ is the boundary of a region [4]. In particular, for any vertex v and any label $M \in \mathscr{B}$, there is a circuit beginning and ending at v with label M and a region whose boundary is this circuit.

Let $\mathscr{U} = \mathbb{S}^2 \backslash \{\mathbf{n},\mathbf{p}\}$. Note that $\mathscr{U}$ is homeomorphic to the punctured plane $\mathbb{R}^2 \backslash \{\mathbf{0}\}$, and we may assume that the circuit ℓ has winding number $+1$ about $\mathbf{0}$.

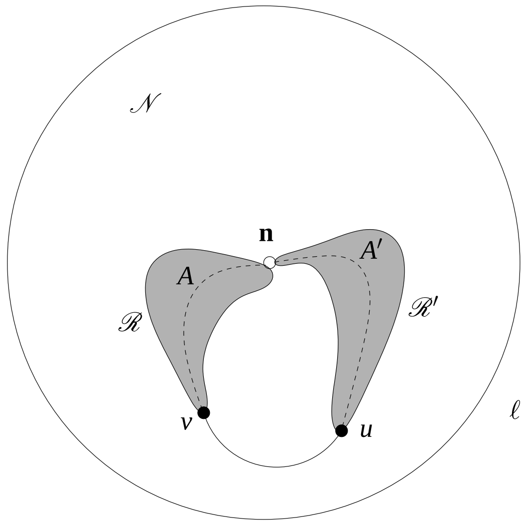

Figure 1:

Let $\widetilde{\mathscr{U}}$ be the universal cover of $\mathscr{U}$ and let $\widetilde{\mathscr{C}}$ be the preimage of $\mathscr{C}$ in $\widetilde{\mathscr{U}}$, where each edge of $\widetilde{\mathscr{C}}$ is oriented and labelled the same way as its image in $\mathscr{C}$. Let $\mathbf{1}$ be the vertex of $\mathscr{C}$ corresponding to the identity element of G, and let $\tilde{\mathbf{1}}$ be a vertex of $\widetilde{\mathscr{C}}$ which lies in the preimage of $\mathbf{1}$.

If M and N are words over X which are congruent mod $\mathscr{B}$, and if v is any vertex of $\mathscr{C}$, then the paths in $\mathscr{C}$ beginning at v and labelled M and N are homotopic in $\mathscr{U}$. Thus, any two paths in $\widetilde{\mathscr{C}}$ beginning at the same vertex and labelled M and N are homotopic in $\widetilde{\mathscr{U}}$; in particular, they have the same endpoint. Conversely, if $\tilde{p}$ and $\tilde{q}$ are two paths in $\widetilde{\mathscr{C}}$ with the same endpoints, then their images in $\mathscr{C}$ are homotopic in $\mathscr{U}$, and so their labels are congruent mod $\mathscr{B}$.

Theorem 2 *$\widetilde{\mathscr{C}}$ is the Cayley graph of the group $\widetilde{G}$ with presentation $\langle X \mid \mathscr{B}\rangle$.*

PROOF. We need to show that $\widetilde{\mathscr{C}}$ is connected, that $\widetilde{G}$ acts on $\widetilde{\mathscr{C}}$ on the left, and that the action is free and transitive on the vertices of $\widetilde{\mathscr{C}}$.

That $\widetilde{\mathscr{C}}$ is connected follows immediately from the facts that $\mathscr{C}$ is connected, and that the preimage $\tilde{\ell}$ of ℓ in $\widetilde{\mathscr{C}}$ is homeomorphic to the real line (and is therefore connected).

We define a left action of $\widetilde{G}$ on $\widetilde{\mathscr{C}}$ as follows: let $\tilde{v}$ be a vertex of $\widetilde{\mathscr{C}}$, and let $g \in \widetilde{G}$. Let M be a word over X representing g and let V be the label of a path in $\widetilde{\mathscr{C}}$ beginning at $\tilde{\mathbf{1}}$ and ending at $\tilde{v}$. We define $g \cdot \tilde{v}$ to be the vertex at the end of the path in $\widetilde{\mathscr{C}}$ beginning at $\tilde{\mathbf{1}}$ and labelled $M \cdot V$ (see Figure 2). This is independent of the choices of M and V: if M' is another word over X representing g, then M' and M are congruent mod $\mathscr{B}$, and hence the paths starting at $\tilde{\mathbf{1}}$ and labelled M and M' (and therefore also those labelled

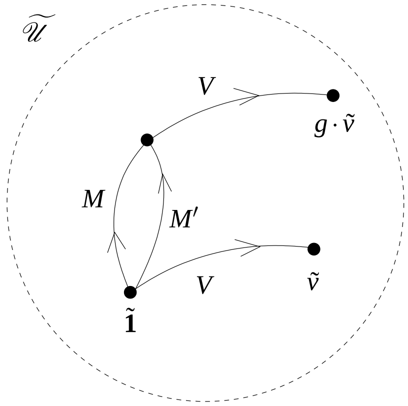

Figure 2:

$M \cdot V$ and $M' \cdot V$) have the same endpoint. A similar argument applies if V' is the label of a different path from $\tilde{\mathbf{1}}$ to $\tilde{v}$.

It is straightforward to verify that this action can be extended to the edges of $\widetilde{\mathscr{C}}$, and so $\widetilde{G}$ acts on $\widetilde{\mathscr{C}}$.

It is obvious that the action is transitive on vertices, since $\widetilde{\mathscr{C}}$ is connected. To see that it is free on vertices, note that if $g \cdot \tilde{v} = \tilde{v}$, then the paths beginning at $\tilde{\mathbf{1}}$ labelled $M \cdot V$ and V are homotopic in $\widetilde{\mathscr{U}}$, which implies that their projections in $\mathscr{C}$ are homotopic in $\mathscr{U}$. But this is only possible if the path labelled M is null-homotopic in $\mathscr{U}$, which implies that M is a product of conjugates of elements of $\mathscr{B}$; that is, that $g = 1$. □

Theorem 3 *Let L be the label of the circuit ℓ. Then $\langle X \mid \mathscr{B} \cup \{L\} \rangle$ is a presentation of G.*

PROOF. Clearly, each word in $\mathscr{B} \cup \{L\}$ represents $\mathbf{1} \in G$. Conversely, if m is any circuit in $\mathscr{C}$ beginning at $\mathbf{1}$, then m is homotopic in $\mathscr{U}$ to ℓ^n for some integer n, and so $m \cdot \ell^{-n}$ is null-homotopic. Thus if M is the label of m, the word ML^{-n} is a product of conjugates of words in $\mathscr{B}$, so that M is a product of conjugates of words in $\mathscr{B} \cup \{L\}$. □

Next we determine the possibilities for the group $\widetilde{G}$ and for the word L.

Since $\widetilde{\mathscr{U}}$ is simply–connected, the union of $\widetilde{\mathscr{C}}$ and all the finite regions in $\widetilde{\mathscr{U}}$ is in fact a Cayley *complex* for $\widetilde{G}$. Furthermore, since each point of the complement $\mathscr{U} \setminus \mathscr{C}$ lies in a finite region, the same is true of each point of $\widetilde{\mathscr{U}} \setminus \widetilde{\mathscr{C}}$. Thus, $\widetilde{\mathscr{U}}$ is homeomorphic to the plane, and so it follows from [6, Chapter III] that $\widetilde{G}$ has a presentation of the following form (see also [3] and [10]):

Generators:

	x_{ij}	$s \geq 0,\ 1 \leq i \leq s,\ r_i \geq 1,\ 1 \leq j \leq r_i$
	e_i	$1 \leq i \leq s$
	c_k	$t \geq 0,\ 1 \leq k \leq t$
	a_p	$g \geq 0,\ 1 \leq p \leq g$
	b_q	either $h = 0$ or $h = g$ and $1 \leq q \leq h$

Relators:

	x_{ij}^2	for all i, j
	$(x_{ij} x_{i\ j+1})^{n_{ij}}$	$1 \leq j < r_i,\ n_{ij} \geq 2$
	$x_{ir_i} e_i x_{i1} e_i^{-1}$	for all i
	$c_k^{\gamma_k}$	$\gamma_k \geq 2$
	$e_1^{-1} \cdots e_s^{-1} c_1^{-1} \cdots c_t^{-1} D$	

where $D = a_1^2 \cdots a_g^2$ if $h = 0$ and $D = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ if $h = g$. Let us call any group with such a presentation a "planar group."

To such a presentation we associate the number

$$\mu = g + h + s - 2 + \sum_{k=1}^{t} \left(1 - \frac{1}{\gamma_k}\right) + \frac{1}{2} \sum_{i,j} \left(1 - \frac{1}{n_{ij}}\right)$$

The following facts about μ can be found in [3]: (1) $\mu < 0$ if and only if G is finite, (2) if $\mu \geq 0$, then its value is independent of which presentation of the above type is used to describe G, and so we may denote it by $\mu(G)$, and (3) if H has finite index in G, then H is also a planar group, and $\mu(H) = [G : H]\mu(G)$.

We observe that the planar groups with $\mu(G) = 0$ are in fact the 17 euclidean plane crystallographic (or "wallpaper") groups. Planar groups with $s = 0$ are called *F-groups* in [6], and the F-groups with $\mu(G) > 0$ are the Fuchsian groups. Finally, those with $s > 0$ and $\mu > 0$ are the (proper) non-euclidean crystallographic groups, or NEC-groups [9].

Recall that the circuit ℓ in $\mathscr{C}$ has label L and winding number $+1$ about $\mathbf{0}$. Since $\mathscr{C}$ is 3–connected, it follows that any circuit of $\mathscr{C}$ with label L has winding number ± 1 about $\mathbf{0}$. Thus, for any word M over X, one of the paths starting at $\mathbf{1}$ and labelled $M^{-1}LML^{-1}$ or $M^{-1}LML$ is null-homotopic, and so the path in $\widetilde{\mathscr{C}}$ with that same label is a circuit. Thus, the element λ of $\widetilde{G}$ represented by the word L has the following property: for any $g \in \widetilde{G}$, either $\lambda^g = g^{-1}\lambda g = \lambda$ or $\lambda^g = \lambda^{-1}$. Of course, λ has infinite order in $\widetilde{G}$.

For any group K, let $S(K) = \{\lambda \in K \mid \lambda$ has infinite order, and $\forall k \in K,$ either $\lambda^k = \lambda$ or $\lambda^k = \lambda^{-1}\}$. Thus if $\lambda \in S(K)$, then either λ belongs to the center of K or there is an element $k \in K$ such that $\lambda^k = \lambda^{-1}$. In the latter case, it is clear that the centralizer of λ in K has index two. That is, any element of $S(K)$ lies either in the center of K or in the center of a subgroup of index two in K.

It follows from [6, Chapter III, Proposition 7.10] that any Fuchsian group has trivial center. Since every subgroup of finite index in a Fuchsian group is also Fuchsian, this implies that $S(K) = \emptyset$ for any Fuchsian group K.

As we shall see, the same is true of proper NEC groups:

Lemma 1 *Let H be a subgroup of finite index in G. If no element of H belongs to $S(G)$, then $S(G) = \emptyset$.*

PROOF. If $g \in S(G)$, then $g^n \in S(G)$ for all $n \in \mathbb{Z}$; since $[G : H]$ is finite, $g^n \in H$ for some n, and so $g^n \in S(G) \cap H$. $\square$

Since obviously $S(G) \cap H \subseteq S(H)$, this implies

Corollary 1 *If $[G : H]$ is finite and $S(H) = \emptyset$, then $S(G) = \emptyset$.*

Finally, any NEC-group has a Fuchsian subgroup of index two [8], and so

Corollary 2 *If G is a proper NEC-group, then $S(G) = \emptyset$.*

Thus, the only possibilities for $\widetilde{G}$ are the 17 wallpaper groups.

Wallpaper groups.

In this section, we will determine $S(W)$ for each of the euclidean plane crystallographic groups W. We refer the reader to [1] for details about these groups, including the notation we will use for them, as well as alternate presentations.

p1 has presentation $\langle a, b \mid [a, b] = 1\rangle$, so **p1** is abelian and each non-identity element has infinite order. Therefore, $S(\mathbf{p1}) = \{a^m b^n \mid (m, n) \neq (0, 0)\}$.

p2 is an extension of **p1** by an element t of order two for which $a^t = a^{-1}$ and $b^t = b^{-1}$. Each element of **p2** can be written uniquely in one of the forms $a^m b^n$ or $a^m b^n t$ for integers m and n.

If $g = a^m b^n$, then $g^t = g^{-1}$. Since $g^a = g^b = g$, all elements $a^m b^n$ with $(m, n) \neq (0, 0)$ belong to $S(G)$.

If $g = a^m b^n t$, then $g^2 = 1$, so $g \notin S(\mathbf{p2})$. Thus, $S(\mathbf{p2}) = \{a^m b^n \mid (m, n) \neq (0, 0)\rangle$.

In terms of the elements $p = tb$, $q = at$, $r = t$ and $u = tba$, **p2** has the presentation $\langle p, q, r, u \mid p^2 = q^2 = r^2 = u^2 = pqru = 1\rangle$ and $S(\mathbf{p2})$ consists of the nontrivial elements of even length.

pm is an extension of **p1** by an element t of order two for which $a^t = a^{-1}$ and $b^t = b$. Again, each element of **pm** has one of the forms $a^m b^n t$ or $a^m b^n$.

If $g = a^m b^n t$, then $g^a = a^{-2} g \neq g$ and $g^a \cdot g = a^{-2} b^{2n} \neq 1$, so $g^a \neq g^{-1}$. Thus, $g \notin S(\mathbf{pm})$.

If $g = a^m b^n$, then $g^t = a^{-m} b^n$, so $g^t = g$ if and only if $m = 0$, and $g^t = g^{-1}$ if and only if $n = 0$. Since g commutes with both a and b, $S(\mathbf{pm}) = \{a^m \mid m \neq 0\} \cup \{b^n \mid n \neq 0\}$.

In terms of the generators b, t and $u = ta$, **pm** has the presentation $\langle b, t, u \mid t^2 = u^2 = 1, bt = tb, bu = ub\rangle$, and $S(\mathbf{pm}) = \{b^n \mid n \neq 0\} \cup \{(tu)^n \mid n \neq 0\}$.

pg has the presentation $\langle x,y \mid y^{-1}xy = x^{-1}\rangle$ (it is the group of the Klein bottle). Each element of **pg** can be written as $x^n y^m$ for unique integers n and m.

If $g = x^m y^n$, then $g^x = x^{m-1}y^n x = x^{m-1}x^{(-1)^n}y^n$, so $g^x = g$ if and only if n is even, and $g^x = g^{-1}$ if and only if $m = n = 0$.

On the other hand, $g^y = y^{-1}x^m y^{n+1} = x^{-m}y^n$, so $g^y = g$ iff $m = 0$ and $g^y = g^{-1}$ iff $n = 0$.

Thus, an element $g = x^m y^n \neq 1$ belongs to $S(\mathbf{pg})$ if and only if either n is even and $m = 0$, or if $n = 0$; that is, iff $g = y^{2k}$ or $g = x^k$ for some integer $k \neq 0$.

In terms of $p = y$ and $q = yx$, **pg** has the presentation $\langle p,q \mid p^2 = q^2\rangle$ and $S(\mathbf{pg}) = \{p^{2n} \mid n \neq 0\} \cup \{(p^{-1}q)^n \mid n \neq 0\}$.

Each of the next three groups—**cm**, **pmg** and **pgg**—is an extension of **pg** by an involution t. For **cm**, $p^t = q$ and $q^t = p$; for **pmg**, $p^t = p^{-1}$ and $q^t = q^{-1}$; and for **pgg**, $p^t = q^{-1}$ and $q^t = p^{-1}$.

If W is any of these three groups, and if $g \in \mathbf{pg} \subseteq W$ belongs to $S(W)$, then $g \in S(\mathbf{pg})$. In all three cases, conjugation by t maps each element of $S(\mathbf{pg})$ either to itself or to its inverse, so $S(\mathbf{pg}) \subseteq S(W)$ in each of them.

Any element of W which belongs to $W \backslash \mathbf{pg}$ is equal to gt for a unique nontrivial $g \in \mathbf{pg}$. Since $(gt)^t = tg$, if $gt \in S(W)$, then either $tg = gt$ or $tg = (gt)^{-1} = tg^{-1}$. In the latter case g has order two, which is impossible since **pg** has no nontrivial elements of finite order. Thus, in all cases, if $gt \in S(W)$, then g is fixed by conjugation by t.

cm: the only elements of **pg** fixed by conjugation by t are p^{2k} for $k \in \mathbb{Z}$. If $k \neq 0$, then $(p^{2k}t)^p = p^{2k-1}qt \neq p^{2k}t$. Also, $(p^{2k}t)^{-1} = p^{-2k}t$, so $(p^{2k}t)^p \neq (p^{2k}t)^{-1}$. Thus, no such element belongs to $S(\mathbf{cm})$; that is, $S(\mathbf{cm}) = S(\mathbf{pg})$.

In terms of t and p, **cm** has the presentation $\langle p,t \mid t^2 = 1, tp^2 = p^2t\rangle$ and $S(\mathbf{cm}) = \{p^{2n} \mid n \neq 0\} \cup \{(p^{-1}tpt)^n \mid n \neq 0\}$.

pmg: conjugation by t fixes only the elements $(p^{-1}q)^k \in \mathbf{pg}$. If $g = (p^{-1}q)^k$ for some $k \neq 0$, then $(gt)^p = p^{-2}(p^{-1}q)^{-k}t \neq gt$. Also, $(gt)^{-1} = (p^{-1}q)^{-k}t \neq (gt)^p$. So again in this case, $S(\mathbf{pmg}) = S(\mathbf{pg})$.

In terms of the generators t, $r = pt$ and $s = qt$, **pmg** has the presentation $\langle t,r,s \mid t^2 = r^2 = s^2 = 1, rtr = sts\rangle$ and $S(\mathbf{pmg}) = \{(rs)^n \mid n \neq 0\} \cup \{(rt)^{2n} \mid n \neq 0\}$.

pgg: no nontrivial element of **pg** is fixed by conjugation by t, so $S(\mathbf{pgg}) = S(\mathbf{pg})$.

In terms of p and $r = pt$, **pgg** has the presentation $\langle p,r \mid (pr)^2 = (p^{-1}r)^2 = 1\rangle$, and $S(\mathbf{pgg}) = \{p^{2n} \mid n \neq 0\} \cup \{r^{2n} \mid n \neq 0\}$.

pmm is the direct product $A \times C$ of the infinite dihedral groups $A = \langle a,b \mid a^2 = b^2 = 1\rangle$ and $C = \langle c,d \mid c^2 = d^2 = 1\rangle$. Note that if $1 \neq \alpha \in A$, then $\alpha^a = \alpha$ iff $\alpha = a$ and $\alpha^b = \alpha$ iff $\alpha = b$. Also, $\alpha^a = \alpha^{-1}$ iff α has even length in terms of a and b, and in this case, $\alpha^b = \alpha^{-1}$, too. Similar statements hold for C.

Let $1 \neq \alpha \in A$ and $1 \neq \gamma \in C$. Then $(\alpha\gamma)^a = \alpha^a\gamma = \alpha\gamma$ if and only if $\alpha = a$, in which case $(\alpha\gamma)^b \neq (\alpha\gamma)^{\pm 1}$, and $(\alpha\gamma)^a = (\alpha\gamma)^{-1} = \alpha^{-1}\gamma^{-1}$ iff the length of α is even and the length of γ is odd.

Similarly, each of $(\alpha\gamma)^c$ and $(\alpha\gamma)^d$ is equal to $(\alpha\gamma)^{\pm 1}$ if and only if the length of α is odd and that of γ is even. Hence, $\alpha\gamma \notin S(\mathbf{pmm})$.

Thus, $S(\mathbf{pmm})$ consists of all the even length elements of A and of C; that is, $S(\mathbf{pmm}) = \{(ab)^n \mid n \neq 0\} \cup \{(cd)^n \mid n \neq 0\}$.

cmm is an extension of **pmm** by an involution t for which $a^t = b$ and $c^t = d$. Note that conjugation by t maps each of A and C onto itself. Also, if $1 \neq \alpha \in A$, then $\alpha^t \neq \alpha$ and $\alpha^t = \alpha^{-1}$ if and only if α has even length in a and b, and similarly for $1 \neq \gamma \in C$.

If $g \in \mathbf{pmm} \subseteq \mathbf{cmm}$ belongs to $S(\mathbf{cmm})$, then $g \in S(\mathbf{pmm})$, so either $g \in A$ and has even length in a and b, or $g \in C$ and g has even length in c and d. In either case, $g^t = g^{-1}$, so $g \in S(\mathbf{cmm})$.

Any other element of $S(\mathbf{cmm})$ has the form $\alpha\gamma t$ for some $\alpha \in A$ and $\gamma \in C$, not both of them 1. Then $(\alpha\gamma t)^a = (a\alpha b)\gamma t \neq \alpha\gamma t$. Also, $(\alpha\gamma t)^{-1} = (\alpha^{-1})^t(\gamma^{-1})^t t = (\alpha\gamma t)^a$ if and only if the length of α is odd and that of γ is even. Similarly, $(\alpha\gamma t)^b = (\alpha\gamma t)^{\pm 1}$ if and only if the length of α is even and that of γ is odd. In short, no element of the form $\alpha\gamma t$ belongs to $S(\mathbf{cmm})$, so $S(\mathbf{cmm}) = S(\mathbf{pmm})$.

In terms of the elements a, c and t, **cmm** has the presentation $\langle a, c, t \mid a^2 = c^2 = t^2 = (ac)^2 = (atct)^2 = 1\rangle$, and $S(\mathbf{cmm}) = \{(at)^{2n} \mid n \neq 0\} \cup \{(ct)^{2n} \mid n \neq 0\}$.

p4 is the extension of **p2** by an element s of order 4 for which $p^s = q$, $q^s = r$, $r^s = u$ and $u^s = p$. In terms of the generators a, b and t of **p2**, $a^s = ba$, $b^s = a^{-2}b^{-1}$ and $t^s = a^{-1}b^{-1}t$.

If $g \in \mathbf{p2}$ belongs to $S(\mathbf{p4})$ then $g \in S(\mathbf{p2})$, so $g = a^m b^n$ for some m and n, at least one of which must be nonzero. But $(a^m b^n)^s = a^{m-2n}b^{m-n}$ which equals $(a^m b^n)^{\pm 1}$ if and only if $m = n = 0$. Thus, no element of **p2** belongs to $S(\mathbf{p4})$. Since $[\mathbf{p4} : \mathbf{p2}] = 4$, Corollary 1 implies that $S(\mathbf{p4}) = \emptyset$.

p3 is an extension of $\mathbf{p1} = \langle a, b \mid [a, b] = 1\rangle$ by an element s of order three for which $a^s = b$ and $b^s = a^{-1}b^{-1}$.

For any m and n, $(a^m b^n)^s = a^{-n}b^{m-n}$, and it is an easy matter to check that this is equal to either $a^m b^n$ or $a^{-m}b^{-n}$ if and only if $m = n = 0$. Thus, no element of **p1** belongs to $S(\mathbf{p3})$. Since **p1** has finite index in **p3**, $S(\mathbf{p3}) = \emptyset$.

Each of the remaining wallpaper groups has either **p3** or **p4** as a subgroup of finite index: **p4** occurs as a subgroup of index two in the groups **p4m** and **p4g**, and **p3** is a subgroup of index two in each of the groups **p3m1**, **p31m** and **p6**, and a subgroup of index four in **p6m**. Thus, $S(W) = \emptyset$ in each of these cases, as well.

Thus, any group with a 3–connected Cayley graph which is embeddable in the sphere with two accumulation points of vertices has one of the following presentations:

- $\langle a, b \mid [a, b] = 1,\ a^n b^m = 1\rangle$ for some $(m, n) \neq (0, 0)$
- $\langle p, q, r, u \mid p^2 = q^2 = r^2 = u^2 = pqru = 1,\ W = 1\rangle$ where W is any word of even length in p, q, r and u.

or, for some integer $n \neq 0$:

- $\langle b,t,u \mid t^2 = u^2 = 1,\, bt = tb,\, bu = ub,\, b^n = 1\rangle$
- $\langle b,t,u \mid t^2 = u^2 = 1,\, bt = tb,\, bu = ub,\, (tu)^n = 1\rangle$
- $\langle p,q \mid p^2 = q^2,\, p^{2n} = 1\rangle$
- $\langle p,q \mid p^2 = q^2,\, (p^{-1}q)^n = 1\rangle$
- $\langle p,t \mid t^2 = 1,\, tp^2 = p^2t,\, p^{2n} = 1\rangle$
- $\langle p,t \mid t^2 = 1,\, tp^2 = p^2t,\, (p^{-1}tpt)^n = 1\rangle$
- $\langle t,r,s \mid t^2 = r^2 = s^2 = 1,\, rtr = sts,\, (rs)^n = 1\rangle$
- $\langle t,r,s \mid t^2 = r^2 = s^2 = 1,\, rtr = sts,\, (rt)^{2n} = 1\rangle$
- $\langle p,r \mid (pr)^2 = (p^{-1}r)^2 = 1,\, p^{2n} = 1\rangle$
- $\langle a,b,c,d \mid a^2 = b^2 = c^2 = d^2 = (ac)^2 = (ad)^2 = (bc)^2 = (bd)^2 = 1,\, (ab)^n = 1\rangle$
- $\langle a,c,t \mid a^2 = c^2 = t^2 = (ac)^2 = (atct)^2 = 1,\, (at)^{2n} = 1\rangle$.

where we have taken account of the fact that in some cases, appending different words in $S(W)$ may give equivalent presentations. We note also that some of these presentations define isomorphic groups.

Finally, it is easy to verify that the Cayley graphs of the groups with these presentations are planar with respect to the given generators, though some of them are not 3–connected.

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