from *"Topology and Combinatorial Group Theory,"*
Springer Lecture Notes in Mathematics, Vol. 1440

The Finite Basis Extension Property and Graph Groups

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INTRODUCTION: A theorem of Marshall Hall, Jr. [5] (cf. also [2], [4]) states that if $B = \{h_1, ..., h_k\}$ is a free basis for a finitely generated subgroup H of a f.g. free group F, and if $\{x_1, \ldots, x_n\}$ is a finite subset of $F - H$, then B can be extended to a free basis for a f.g. subgroup H^* of finite index in F such that $\{x_1, \ldots, x_n\} \subset F - H^*$.

A similar theorem holds for free abelian groups, and this may be regarded as a generalization of the fact that in a vector space, every linearly independent set can be extended to a basis.

We will prove that an analogous *basis extension property* holds in a class of groups which contains the f.g. free and free abelian groups.

Given a graph $\Gamma = (V, E)$, let F_Γ denote the group with presentation

$$\langle V \mid \{xy = yx : (x, y) \in E\}\rangle$$

Any group G isomorphic to F_Γ for some Γ is called a *graph group*, and the image of the vertex set V under any isomorphism $F_\Gamma \to G$ will be called a *basis* for G. Note that free groups and free abelian groups are graph groups, and for these groups this corresponds to the usual notion of basis.

A graph group F_Γ is said to have the *finite basis extension property*, FBEP for short, if F_Γ satisfies the following analog of Marshall Hall's property: given any f.g. subgroup H of F_Γ such that H is itself a graph group, and given a finite set $\{x_1, \ldots, x_n\} \subset F_\Gamma - H$, then H has a basis which can be extended to a basis for a subgroup H^* of finite index in F_Γ such that $\{x_1, \ldots, x_n\} \subset F - H^*$.

We remark that not every subgroup of a graph group need have a basis, that is, not every subgroup of a graph group need be a graph group. Indeed, the following are equivalent [3]:

1. every f.g. subgroup of F_Γ is a graph group

2. Γ has no full subgraph isomorphic to either a square or the three edge path, ○—○—○—○

3. F_Γ belongs to the smallest collection of groups containing the infinite cyclic group $\mathbb{Z}$, and closed under the binary operation $(-)*(-)$ (free product) and unary operation $\mathbb{Z}\oplus(-)$ (direct product with $\mathbb{Z}$.)

If Γ satisies 2, Γ is called a *special assembly.*

Our main result is:

THEOREM 1 *Let Γ be a finite graph. Then F_Γ has FBEP if and only if Γ is a special assembly.*

A group has the *finitely generated intersection property*, FGIP, if the intesection of any two of its f.g. subgroups is f.g. Howson [6] proved that free groups have FGIP. In section 4 we will show that a graph group F_Γ has FGIP iff every component of Γ is complete.

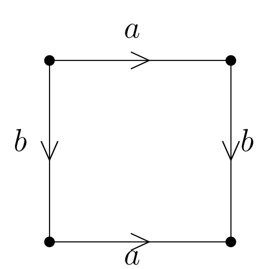
THEOREM 1 - SUFFICIENCY: Let $A=(V,E)$ be a finite special assembly. For each $a\in F_\Gamma$, we define $|a|_A$, the *A-length* of a, to be the length of the shortest word in $V^{\pm 1}$ which represents a. If it is clear which graph is meant, we will often refer to the *length*, $|a|$, of a.

PROPOSITION 1 *Let H be a f.g. subgroup of F_A and let $M\geq 0$. Then H has a basis which can be extended to a basis for a subgroup $H^*\leq F_A$, with $[F_A:H^*]<\infty$, and such that if $x\in H^*$ and $|x|_A<M$, then $x\in H$. In particular, F_A has FBEP.*

The length condition has the following topological interpretation: let C_A be the Cayley complex of the presentation

$$\langle V \mid \{xy=yx:(x,y)\in E\}\rangle$$

That is, C_A has one 0-cell, an oriented 1-cell for each vertex in V, and for each edge $(a,b)\in E$, a 2-cell attached by



Words in $V^{\pm 1}$ correspond to cellular loops in C_A, and the length of a word is equal to the total number of edges traversed, counting multiplicity. For subgroups $H\leq H^*\leq F_A$, take covers $X_H\to X_{H^*}\to C_A$, with base points v_H and v_{H^*} realizing these subgroups, giving X_H and X_{H^*} the induced cell decompositions. Let $k>0$, and define $[X_H]_k$ to be the subcomplex of X_H consisting of those vertices which are joined to the basepoint by a cellular path of length $<k$, together with all 1 and 2-cells spanned by those vertices. Elementary covering space theory gives

Proposition 2 *The following are equivalent:*

1. *For all $x \in H^*$, if $|x| < 2M$ then $x \in H$,*

2. *The covering map $X_H \to X_{H^*}$ maps $[X_H]_M$ homeomorphically onto $[X_{H^*}]_M$.*

Proof: (of Proposition 1) We assume that F_A is not infinite cyclic, so there are two cases: either A is disconnected, so $F_A = F_{A_1} * F_{A_2}$ or $F_A = \mathbb{Z} \oplus F_{A_1}$, for smaller special assemblies A_1 and A_2.

Case 1 Realize each F_{A_i} by its Cayley complex, $\mathcal{A}_i$, with vertex p_i, as described before. F_A is then realized by attaching p_1 and p_2 to a third vertex, p, by edges e_1 and e_2 respectively, as shown below: Denote this complex by $\mathcal{F}$, and take p to be the

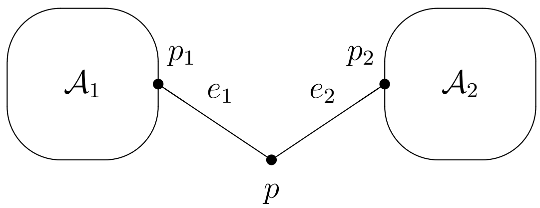


Figure 1: The complex $\mathcal{F}$

base point. Realize $H \leq F_A$ by a cover $\mathcal{H}$ with base point p_H covering p. $\mathcal{A}_i$ lifts to a disjoint union of covers $\mathcal{A}_i^{(j)}$ of $\mathcal{A}_i$ in $\mathcal{H}$. H is finitely generated, and since it is a graph group, it is finitely presented as well, so $\pi_1(\mathcal{H}, p_H)$ is carried by some finite subcomplex of $\mathcal{H}$. Thus, there is some $M' \geq M$ such that $\pi_1([\mathcal{H}]_{M'}, p_H) = \pi_1(\mathcal{H}, p_H)$. Let $\mathcal{U}$ denote the union of $[\mathcal{H}]_{M'}$ and all those $\mathcal{A}_i^{(j)}$ which intersect $[\mathcal{F}_H]_{M'}$ non-trivially, noting that there are only finitely many such $\mathcal{A}_i^{(j)}$'s. Then

$$H = \pi_1(\mathcal{H}, p_H) = \pi_1([\mathcal{H}]_{M'}, p_H) = \pi_1(\mathcal{U}, p_H).$$

Now, choose a maximal tree for each $\mathcal{A}_i^{(j)}$, their union is a forest for $\mathcal{U}$. Extend this to a maximal tree T of $\mathcal{U}$ by adding only lifts of the edges e_1 and e_2. Choose a base point p_{ij} for each $\mathcal{A}_i^{(j)}$ and a path w_{ij} in T from the base point to p_{ij}.

For each $\mathcal{A}_i^{(j)}$ contained in $\mathcal{U}$, $\pi_1(\mathcal{A}_i^{(j)}, p_{ij})$ is a finitely generated special assembly group, so, by induction, $\mathcal{A}_i^{(j)}$ has a cover $\mathcal{B}_i^{(j)}$ such that $\pi_1(\mathcal{B}_i^{(j)})$ has a basis $B_i^{(j)}$ which is an extension of a basis $A_i^{(j)}$ of $\pi_1(\mathcal{A}_i^{(j)})$, and such that $[\mathcal{B}_i^{(j)}]_{2M'} = [\mathcal{A}_i^{(j)}]_{2M'}$. $\pi_1(\mathcal{U}, p)$ has a basis A', (see [9], page 167), whose connected components are $w_{ij} A_i^{(j)} w_{ij}^{-1}$, together with a set of isolated vertices, E, corresponding to lifts of e_1 and e_2 which are not in T. If we replace each $\mathcal{A}_i^{(j)}$ with $\mathcal{B}_i^{(j)}$ along $[\mathcal{A}_i^{(j)}]_{2M}$ to form $\mathcal{U}'$, then $\pi_1(\mathcal{U}', p)$ has a basis B whose connected components are $w_{ij} B_i^{(j)} w_{ij}^{-1}$, together with E. So B is an extension of A', and $[\mathcal{U}']_{2M} = [\mathcal{F}_H]_{2M}$.

$\mathcal{U}'$ is a finite complex, but it is not a cover, since there may be a lift of p_i which is not adjacent to a lift of e_i. At each such vertex attach a copy of $\mathcal{F} - \mathcal{A}_i$, to form $\mathcal{H}^*$. $\mathcal{H}^*$ is a finite cover of $\mathcal{F}$, and $\pi_1(\mathcal{H}^*, p)$ has a basis B', which is the union of B with some connected components isomorphic to either A_1 or A_2, hence B' is an extension of A', and since $[\mathcal{H}^*]_{2M} = [\mathcal{F}_H]_{2M}$, the length condition is also satisfied.

Case 2 $F_A = \langle t \rangle \oplus F_{A_1}$, where A_1 is a smaller special assembly. Let $M \geq 0$ be given, and let $H \leq F_A$ be finitely generated. Then there is an exact sequence

$$1 \to H \cap \langle t \rangle \to H \to \rho(H) \to 1$$

where $\rho : F_A \to F_{A_1}$ is the natural projection. Now, $\rho(H)$ is finitely generated, so there is a subgroup K of finite index in F_{A_1}, with a basis Y such that:

1. some subset X of Y is a basis for $\rho(H)$, and
2. if $k \in K$ has A-length $\leq M$, then $k \in \rho(H)$.

Let X^* be any preimage of X in H, let Y^* be any preimage of Y containing X^*, and let K^* be the group generated by Y^*.

Suppose first that $H \cap \langle t \rangle = \langle t^k \rangle$, for some $k > 0$. Then the set $\{t^k\} \cup X^*$ is a basis for H, [3]. Clearly $\{t^k\} \cup Y^*$ is a basis for $\langle t^k \rangle \oplus K = K^*$, and this set contains a basis for H. Let $k \in K^*$ have A-length $\leq M$. Then $\rho(k)$ has A_1-length $\leq M$, and so $\rho(k) \in \rho(H)$. Thus, $k \in \mathrm{gp}\langle t^k, X^* \rangle = H$.

Now suppose that $H \cap \langle t \rangle = \{1\}$. Let $X^* = \{t^{n_i} x_i\}$, where each $x_i \in F_{A_1}$, and let $N = \max|n_i|$. Since F_{A_1} contains only finitely many elements of length $\leq M$, we can choose T such that any reduced product of T or more x_i's and their inverses has length $> M$. Let $L = M + TN + 1$, and let $K^* = K \otimes \langle t^L \rangle$. Then, as before, if $k \in K^*$ has A-length $\leq M$, then $\rho(k) \in \rho(H)$, so that $k \in \mathrm{gp}\langle t^L, X^* \rangle$. Suppose $k = (t^L)^s (t^{m_1} y_1 t^{m_2} y_2 \cdots t^{m_r} y_r)$, where each $t^{m_i} y_i = (t^{n_j} x_j)^{\pm 1}$, for some j. Suppose s $\neq 0$. Then $|k|_A = |sL + \sum m_i| + |y_1 y_2 \cdots y_r|_A$, and, since $|k|_A \leq M$, $r < T$. Therefore, $|k|_A \geq |s|L - |\sum m_i| \geq |s|L - \sum |m_i| \geq |s|L - TN \geq L - TN = M + 1$, a contradiction. Therefore, $s = 0$, and $k \in H$.

□

THEOREM 1 - NECESSITY:

PROPOSITION 3 *If Γ is finite and F_Γ has FBEP, then Γ is a special assembly.*

We prove this via a sequence of lemmas:

LEMMA 1 *If Γ is a connected graph, then F_Γ is (freely) indecomposable, that is, F_Γ is not the free product of two of its nontrivial subgroups.*

PROOF: This is clear if F_Γ is infinite cyclic, so suppose Γ has more than one vertex, and that $F_\Gamma = G * H$. Let v be a vertex of Γ. Then v is adjacent to at least one other vertex of Γ, so the centralizer of v in F_Γ is not cyclic. Therefore, v belongs to a conjugate either of G or of H, so we may suppose that $v \in G$. But then any element which commutes with v must also lie in G, in particular, any vertex adjacent to v must lie in G. Thus, since any vertex of Γ can be reached by a path from v, G must contain all the vertices of Γ. But the vertices of Γ generate F_Γ, so $H = 1$. □

LEMMA 2 *Let* Γ *be finite graph such that* F_Γ *has FBEP, and suppose that* Γ *and its complement are both connected. Then* Γ *consists of a single vertex. (In particular, any graph group which has FBEP is either infinite cyclic, or it is the free or the direct product of two of its subgroups.)*

PROOF: Suppose Γ has more than one vertex. Then F_Γ contains a free abelian subgroup of rank 2, and is indecomposable, by Lemma 1. Thus, if H is a subgroup of finite index in F_Γ, then H is not infinite cyclic, and H is indecomposable. Let $v_1, \ldots, v_k$ be the vertices of Γ, and let H be the infinite cyclic subgroup generated by the element $v = v_1 \cdots v_k$. Suppose H^* has finite index in F_Γ, and suppose H^* has a basis B containing v. Since Γ has connected complement, the centralizer of v in F_Γ is cyclic, [7]. Thus, v does not commute with any other elements of B. Thus, either H is a proper free factor of H^*, or $H^* = H$, neither of which is possible. So Γ has only one vertex. □

LEMMA 3 *If* $G_1 \oplus G_2$ *is the direct product of two f.g. graph groups and* $G_1 \oplus G_2$ *has FBEP, then each of* G_1 *and* G_2 *has FBEP.*

PROOF: Let A be a basis for $H \leq G_1$, and let C_2 be a basis for G_2. Then $A \cup C_2$ is a basis for a subgroup $H \oplus G_2 \leq G_1 \oplus G_2$. Since $G_1 \oplus G_2$ has FBEP, there is a set D such that $A \cup C_2 \cup D$ is a basis for a subgroup H^* of finite index in $G_1 \oplus G_2$. Let $p : G_1 \oplus G_2 \to G_1$ denote the natural projection. Then $p : \langle A \cup D \rangle \to G_1$ is injective, since the sets $A \cup D$ and C_2 generate subgroups with trivial intersection. Thus $p(A \cup D) = A \cup p(D)$ is a basis for a subgroup K^* of G_1. Moreover, $H^* = K^* \oplus G_2$, and since H^* has finite index in $G_1 \oplus G_2$, K^* has finite index in G_1. □

LEMMA 4 *Let* $A = A_1 * \cdots * A_k$ *be the free product of f.g. graph groups, where the underlying graphs of the free factors are connected. If* A *has FBEP, then each* A_i *does, as well.*

PROOF: Let B_1 be a finite basis for a subgroup H_1 of A_1, and let $B = B_1 \cup B_2$ be a basis for a subgroup H^* of finite index in A. Now, B is the set of vertices of a graph Γ; let C denote the set of all vertices which can be reached by a path in Γ from some vertex in B_1. Clearly, $B_1 \subset C$, and by an argument similar to that in Lemma 1, $C \subset A_1$. Let H_1^* be the group generated by C. To show that A_1 has FBEP, it will suffice to show that H_1^* has finite index in A_1. C is a union of connected components of Γ, so H_1^* is a free factor of H^*. Thus, $H_1^* \cap A_1 = H_1^*$ is a free factor of $H^* \cap A_1$, and $H^* \cap A_1$ has finite index in A_1. But A_1 is indecomposable, so $H^* \cap A_1$ is indecomposable. Therefore, $H_1^* = H^* \cap A_1$. Thus H_1^* has finite index in A_1, so A_1 has FBEP. □

We have shown so far that if F_Γ has FBEP, then it belongs to the smallest class of groups containing the integers which is closed with respect to free products and direct products. We remark that F_Γ belongs to this class if and only if Γ has no full subgraphs isomorphic to ∘— ∘ — ∘ —∘, such a graph is called an *assembly.* To finish the proof of Proposition 3, it will therefore suffice to show:

LEMMA 5 *If* Γ *is finite and* F_Γ *has FBEP, then* Γ *has no full subgraph isomorphic to the square.*

PROOF: Since the square is connected, we may suppose that Γ is connected. If Γ contains a square, then F_Γ has a direct factor of the form $A = (A_1 * A_2) \oplus (A_3 * A_4)$, where each $A_i = F_{\Gamma_i}$ for some assembly F_{Γ_i}. If F_Γ has FBEP, then, by the above, so does A. Consider an element $a = a_1a_2a_3a_4$, where a_i is a vertex of Γ_i. We will show that the set $\{a\}$ cannot be extended to a basis for a subgroup of finite index in $(A_1 * A_2) \otimes (A_3 * A_4)$. Suppose $H^* \equiv F_\Sigma$ has finite index in A, and that a is a vertex of Σ. Then Σ must be an assembly, [8], and it must be connected since A is indecomposable. By [7], the centralizer in A of a is the subgroup $\langle a_1a_2\rangle \oplus \langle a_3a_4\rangle$, which is a free abelian group of rank two. So if a is a vertex of Σ, it must be a pendant vertex. But a connected assembly with a pendant vertex is a star, so $H^* \equiv F \oplus \mathbb{Z}$, where F is free. Thus, any subgroup of H^* is either free or the direct product of $\mathbb{Z}$ with a free group [3]. Now, A contains a subgroup K isomorphic to $F_2 \oplus F_2$, where F_2 is free of rank 2, and $H^* \cap K$ has finite index in K. But it is straightforward to see that any subgroup of finite index in K contains a subgroup isomorphic to $F_n \oplus F_m$, where F_n and F_m are non-cyclic free groups. This is impossible, since $H^* \cap K \leq H^*$. Thus, A does not have FBEP. □

THE FINITELY-GENERATED INTERSECTION PROPERTY:

THEOREM 2 *The graph group F_Γ has FGIP iff each connected component of Γ is a complete graph.*

PROOF: The given condition is equivalent to requiring that no full subgraph of Γ be isomorphic to L_2,

$$L_2 = \overset{x}{\circ} \text{——} \overset{y}{\circ} \text{——} \overset{z}{\circ}\ .$$

Suppose first that every connected component of X is a complete graph. Then F_X is either a free abelian group, or it is a free product of free abelian groups, which has FGIP by [1].

For the converse, it will suffice to show that the group F_{L_2} does not have FGIP, since any subgroup of a group with FGIP must itself have FGIP. Let H be the subgroup of F_{L_2} generated by the elements $x^{-1}y$ and $y^{-1}z$. Let t be a generator of an infinite cyclic group. Then H is the kernel of the homomorphism $f : F_{L_2} \to \langle t\rangle$ defined by $f(x) = f(y) = f(z) = t$. Let K be the subgroup of F_{L_2} generated by x and z. Clearly K is free. Now H and K are both finitely generated, but their intersection is the kernel of the restriction of f to K; since this kernel is the normal closure in K of the element $x^{-1}z$, it is free of infinite rank. Thus, F_{L_2} does not have FGIP. □

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