Infinite–ended groups with planar Cayley graphs

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Abstract

We find necessary and sufficient conditions for a finitely generated group with more than one end to have a planar Cayley graph.

1 Introduction

Given a group G and a set X of generators of G, the Cayley graph $\mathcal{C}(G,X)$ has vertex set G, and for each $g \in G$ and $x \in X$, an edge directed from g to gx and labelled "x." If $x \in X$ has order two, then for each $g \in G$, we replace the edges labelled x directed from g to gx and from gx to g by a single undirected edge joining g and gx and also labelled x.

A graph is called *planar* if it can be drawn in the 2-sphere $\mathbb{S}^2$ (or, equivalently, in the plane) in such a way that the interiors of different edges are disjoint. The finite groups which have planar Cayley graphs were enumerated by Maschke in 1896 [13]—they are just the finite subgroups of the full symmetry group of $\mathbb{S}^2$.

Finitely generated infinite groups whose Cayley graphs can be embedded in the 2–sphere with a single accumulation point of vertices have also been enumerated, by Zieschang, Vogt and Coldewey [19] and Wilkie [18]. For such an embedding, the union of the embedded graph and the regions whose boundaries are finite circuits of the graph is a Cayley *complex* for the group; in these cases, if the group is one–ended, then it is either a Fuchsian group or a non–euclidean crystallographic group. (See also chapter 3 of [12].)

If the Cayley graph of an infinite, finitely generated group can be embedded in the sphere, but only with two or more accumulation points of vertices, then the group has more than one end [9, Theorem 3]. In this case, a celebrated theorem of Stallings [15] asserts that the group is either a free product with finite amalgamated subgroups or an HNN extension with finite associated subgroups.

In this article, we find necessary and sufficient conditions for such a group to have a planar Cayley graph. In addition, we show that any finitely generated group with a planar Cayley graph is accessible. Since the one-ended ones have been determined, it is possible that these groups can be effectively enumerated, though we do not yet know if this is the case.

2 Sufficient conditions

In this section, we will describe certain somewhat technical sets of conditions which guarantee that a free product with finite amalgamations or an HNN extension with finite associated subgroups possesses a planar Cayley graph.

Given a 5-tuple (G,A,H,B,ϕ), where G and H are groups with subgroups A and B, respectively, and $\phi : A \to B$ is an isomorphism, the *free product with amalgamations* is the group $\mathrm{FPA}(G,A,H,B,\phi) = (G*H)/N$, where N is the normal closure in $G*H$ of the set $\{a^{-1}\phi(a) \mid a \in A\}$. G and H are called the *base groups* and A and B are called the *amalgamated subgroups.* Given a 4-tuple (G,A,B,ψ), where G is a group, A and B are subgroups of G, and $\psi : A \to B$ is an isomorphism, the *HNN extension* is the group $\mathrm{HNN}(G,A,B,\psi) = (G*\mathbb{Z})/N$, where N is the normal closure in $G*\mathbb{Z}$ of the set $\{t^{-1}a^{-1}t\psi(a) \mid a \in A\}$ and t is a generator of the infinite cyclic factor $\mathbb{Z}$. G is called the *base group* and A and B the *associated subgroups* of the HNN extension.

Suppose G is a group with a finite subgroup A, and suppose some Cayley graph of G has an embedding $\mathcal{G}$ in the 2–sphere $\mathbb{S}^2$ with the following properties:

1. for each left coset gA of A in G, there is an open disk $D_{\mathcal{G}}(gA)$ in $\mathbb{S}^2$ which is disjoint from $\mathcal{G}$, and whose boundary $\partial D_{\mathcal{G}}(gA)$ contains the elements of gA and is otherwise disjoint from $\mathcal{G}$.

2. the disks $D_{\mathcal{G}}(gA)$ and $D_{\mathcal{G}}(g'A)$ are disjoint if $gA \neq g'A$

3. for any $g \in G$, the cyclic order of the vertices in $\partial D_{\mathcal{G}}(A)$ coincides up to orientation with that of their images under left multiplication by g in $\partial D_{\mathcal{G}}(gA)$

Then we will call $(\mathcal{G},A)$ a *planar pair.* In particular, note that if A is trivial, then these three conditions are satisfied by any embedding of any Cayley graph of G in $\mathbb{S}^2$, and the same is true if A has order two, provided the non-trivial element of A belongs to the set of generators of G which define $\mathcal{G}$. Note also that the third condition implies that A acts on an $|A|$–gon whose vertices are the elements of A; that is, A is a finite cyclic or dihedral group.

Now suppose that A is a subgroup of G and that B is a subgroup of H. Further suppose that G and H have Cayley graphs with embeddings $\mathcal{G}$ and $\mathcal{H}$, respectively, in $\mathbb{S}^2$ such that $(\mathcal{G},A)$ and $(\mathcal{H},B)$ are both planar pairs. Finally, suppose that $\phi : A \to B$ is an isomorphism, and that the cyclic order of the vertices on $\partial D_{\mathcal{G}}(A)$ agrees up to orientation with that of their images under ϕ on $\partial D_{\mathcal{H}}(B)$. Then we will call (G,A,H,B,ϕ) a *planar 5-tuple.*

Example 2.1 ([4]) Consider the Coxeter groups

$$G = \langle x,y,z \mid x^2 = y^2 = z^2 = (xy)^2 = (yz)^2 = (zx)^2 = 1\rangle$$

and

$$H = \langle a,b,c \mid a^2 = b^2 = c^2 = (ab)^2 = (bc)^2 = (ca)^3 = 1\rangle$$

and let $\mathcal{G}$ and $\mathcal{H}$ be the usual embeddings of their Cayley graphs with respect to the given generators in $\mathbb{S}^2$. $\mathcal{G}$ can be thought of as the 1–skeleton of a 3–cube and $\mathcal{H}$ as that of a hexagonal cylinder (see figure 1.) Let A be the subgroup of G generated by

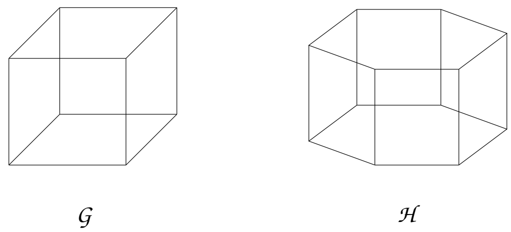

Figure 1: Cayley graphs of G and H

$\{x, y\}$ and let B be the subgroup of H generated by $\{a, b\}$. Define $\phi : A \to B$ by $\phi(x) = a$ and $\phi(y) = b$. Then it is straightforward to verify that (G, A, H, B, ϕ) is a planar 5-tuple.

Let $L = \text{FPA}(G, A, H, B, \phi)$. Then L is generated by $\{x, y, z, c\}$. We can build an embedding $\mathcal{L}$ of the Cayley graph of L with respect to these generators in $\mathbb{S}^2$ by the following iterative construction: begin with a copy of $\mathcal{G}$. Then to each face of $\mathcal{G}$ corresponding to a left coset of A, attach a copy of $\mathcal{H}$ along its *unit* left coset of B (ie, the coset $1 \cdot B$.) Next, attach to each *new* non-unit left coset of B a copy of $\mathcal{G}$ along its unit left coset of A, and repeat these steps (see figure 2.) Note that the portion of $\mathcal{L}$

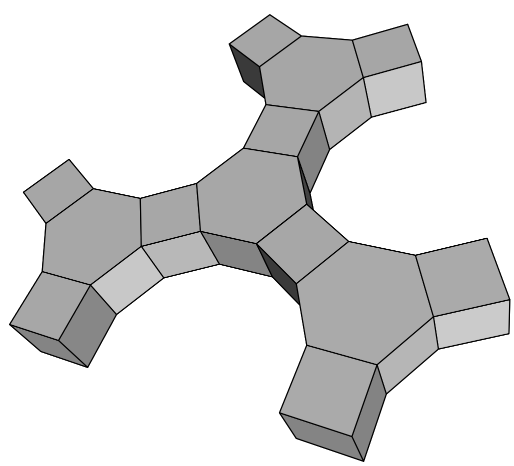

Figure 2: A portion of the Cayley graph of $\text{FPA}(G, A, H, B, \phi)$

constructed after each finite stage can be embedded in the connected sum of 2–spheres, which is to say, in the 2–sphere itself, and so $\mathcal{L}$, which is the limit of these, is also planar. □

Next suppose G is a group with two finite subgroups A and B which are isomorphic via a map $\psi : A \to B$. We will call (G, A, B, ψ) a *planar quadruple* if some Cayley graph of G has an embedding $\mathcal{G}$ in $\mathbb{S}^2$ with these properties:

1. $(\mathcal{G}, A)$ and $(\mathcal{G}, B)$ are both planar pairs.
2. $D_{\mathcal{G}}(gA)$ is disjoint from $D_{\mathcal{G}}(g'B)$ for all $g, g' \in G$.
3. The cyclic order of the vertices around $\partial D_{\mathcal{G}}(A)$ coincides up to orientation with that of their images under ψ around $\partial D_{\mathcal{G}}(B)$.

Once again, these conditions are satisfied by any embedding of the Cayley graph of G in $\mathbb{S}^2$ if A and B are trivial, or if they both have order two, and both nontrivial elements belong to the generating set used to define $\mathcal{G}$. Also, as before, these conditions imply that A and B are finite cyclic or dihedral groups.

Example 2.2 Let G and A be as in the previous example, and let B be the subgroup of G generated by $\{y, z\}$. Define $\psi : A \to B$ by $\psi(x) = y$ and $\psi(y) = z$. It is again straightforward to see that (G, A, B, ψ) is a planar quadruple.

Let $K = \mathrm{HNN}(G, A, B, \psi)$. Then K is generated by $\{x, y, z, t\}$, and an embedding $\mathcal{K}$ in $\mathbb{S}^2$ of its Cayley graph with respect to these generators can be built by an iterative procedure similar to the one above. Beginning with a copy of $\mathcal{G}$, we attach to each face of $\mathcal{G}$ corresponding to a left coset of A another copy of $\mathcal{G}$ joining the vertices in the given coset of A in the old copy to the vertices of the unit left coset of B in the new with edges labelled "t", and directed from the old copy of $\mathcal{G}$ to the new. Similarly, to each left coset of B we attach a new copy of $\mathcal{G}$, with edges labelled "t" joining the vertices of the unit left coset of A to the vertices of the given left coset of B, this time directed from new to old. Then we repeat these constructions with each new coset of A and of B that is added (see figure 3.) As before, we see that the graph produced at each finite stage of the construction is planar, and so $\mathcal{K}$ is, as well. □

In fact, it is easy to see that the procedures of the above examples can be applied to produce embeddings in $\mathbb{S}^2$ of a Cayley graph of $\mathrm{FPA}(G, A, H, B, \phi)$ for any planar quintuple (G, A, H, B, ϕ), and of one for $\mathrm{HNN}(G, A, B, \psi)$ for any planar quadruple (G, A, B, ψ).

The purpose of this note is to prove the converse of this statement; that is, any finitely-generated group with more than one end which has a planar Cayley graph is either an amalgamated free product or an HNN extension of one of the above kinds.

3 Groups with more than one end

Most of the ideas in this section had their origins in the work of Stallings [15, 16]; our formulation is taken primarily from [5], [6] and [14] (see also [2].) Let G be a finitely generated group, and let X be a finite generating set for G. Let $\mathcal{C}(G, X)$ denote

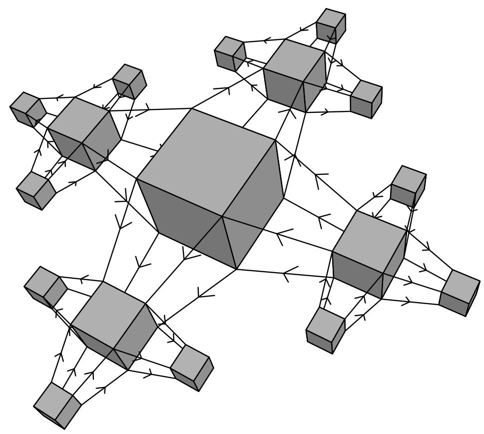

Figure 3: A portion of the Cayley graph of $\mathrm{HNN}(G,A,B,\psi)$

the Cayley graph of G with respect to X. If X' is any other finite generating set for G, then the number of ends of $\mathcal{C}(G,X)$ is the same as that of $\mathcal{C}(G,X')$, and so we refer to this number as the number of ends of G. It is well-known that the number of ends of a finitely generated group is either 0 (in case G is finite), 1 (for example, if $G \simeq \mathbb{Z}\times\mathbb{Z}$), 2 (if G is virtually infinite cyclic) or uncountably infinite (if G is free of rank two, for example.)

Stallings's Theorem [15, 16] states that any finitely generated group with more than one end is either a nontrivial free product with finite amalgamated subgroups, or an HNN extension with finite associated subgroups—that is, G is the fundamental group of a graph of groups which has one edge, where the edge group is finite.

Given a subgraph f of a graph Γ, we define $\overline{f}$ to be the subgraph of Γ spanned by the vertices which do not belong to f. We call $\overline{f}$ the *complement* of f. The set of edges which belong neither to f nor to $\overline{f}$ (that is, the set of edges which have one endpoint in f and the other in $\overline{f}$) is called the *coboundary* of f, and is denoted δf. Note that $\delta f = \delta\overline{f}$.

Let $\mathcal{C}$ be a Cayley graph for G, and suppose that G (and hence $\mathcal{C}$) has more than one end. Then there is a *cut* in $\mathcal{C}$: that is, an infinite connected subgraph e_0 whose complement $\overline{e_0}$ is also connected and infinite, and whose coboundary is finite. Let $\widetilde{E} = \{ge_0 \mid g \in G\} \cup \{g\overline{e_0} \mid g \in G\}$. Then there is an equivalence relation on $\widetilde{E}$, defined as follows: given x and y in $\widetilde{E}$, we set $x \approx y$ if x is, among elements of $\widetilde{E}$, a maximal *proper* subset of $\overline{y}$. (Note that $x \not\approx \overline{x}$ for all x, and that if $x \approx y$, then δx is disjoint from y.) Let $\widetilde{V}$ denote the set of $\approx$–classes.

Lemma 3.1 *[5, 6] The sets $\widetilde{V}$ and $\widetilde{E}$ are, respectively, the vertex and edge sets of an undirected tree $\widetilde{T}$, on which G acts. Each pair $\{x,\overline{x}\} \subseteq \widetilde{E}$ represents the two possible* *orientations of an edge of $\widetilde{T}$. The initial vertex of any (oriented) edge $x \in \widetilde{E}$ is $[x] \in \widetilde{V}$, and its terminal vertex is $[\overline{x}]$.*

Next, we observe that G also acts on a *directed* tree. If e_0 and $\overline{e_0}$ lie in different orbits under the G–action on $\widetilde{E}$, then we define $V = \widetilde{V}$, $E = \{ge_0 \mid g \in G\}$, and T is the tree with vertex set V and edge set E. Suppose, on the other hand, that there is an element $t \in G$ such that $te_0 = \overline{e_0}$ (ie, t reverses the undirected edge $\{e_0, \overline{e_0}\}$.) Then each undirected edge of $\widetilde{T}$ is reversed by some element of G. In this case, we *subdivide* the edges of $\widetilde{T}$; formally, we let $V = \widetilde{V} \cup \{\{ge_0, \overline{ge_0}\} \mid g \in G\}$ (that is, we add a new vertex $\{ge_0, g\overline{e_0}\}$ for each undirected edge of $\widetilde{T}$), and we let $E = \widetilde{E}$. For any $x \in E$, we let $[x] \in V$ be the initial vertex of x, and we let $\{x, \overline{x}\} \in V$ be its terminal vertex (see figure 4.) In either case, T, with vertex set V and edge set E, is a directed tree, and G acts on T with one orbit of edges.

Figure 4: Subdividing an edge

4 Planar Cayley graphs

Now suppose G is a group with a Cayley graph that is planar and has more than one end, and let $\mathcal{G}$ be an embedding of this graph in $\mathbb{S}^2$. Then there is a cut e_0 in $\mathcal{G}$, and a directed tree T as above on which G acts with one orbit of edges. Let v be a vertex of T. Then v is a set of G–translates of e_0 and $\overline{e_0}$, which may consist of a single pair $\{ge_0, g\overline{e_0}\}$ in the case of a vertex added during the "subdivision" process discussed above. Thus, $\mathcal{G} = (\cup_{x \in v} x) \cup P$, where P consists of all the vertices and edges of $\mathcal{G}$ that belong to no $x \in v$—in particular, P contains the edges in δx for each $x \in v$ (see figure 5.) Let $e \in v$ (so that v is the initial vertex of e and the terminal vertex of $\overline{e}$.) Let G_v be the G–stabilizer of v under the action of G on T. If $g \in G_v$, then g permutes the elements $x \in v$ setwise among themselves, and $gP = P$.

Suppose for the moment that $\mathcal{G}$ is three–connected. Then it follows from the infinite version of Whitney's theorem [8] that if $x, y \in v$ belong to the same G–orbit, then the cyclic orders of the edges in their respective coboundaries agree, up to orientation. Thus, we may modify $\mathcal{G}$ by replacing each subgraph $x \in v$ with a circuit C_x whose length is $|\delta x|$, and in this way we obtain a new graph (which we will call $\mathcal{G}_v$) on which G acts (see figure 6.) Note that $\mathcal{G}_v$ is also three–connected.

Let a be any vertex of C_e. Since G acts freely on the vertices and edges of $\mathcal{G}$, it follows that if $1 \neq g \in G_v$, then g can fix no vertex of $\mathcal{G}_v$. Therefore, $\mathcal{G}_v$ can be contracted onto a graph $\widetilde{\mathcal{G}_v}$ which is a Cayley graph for G_v, and in which the image of a corresponds to $1 \in G_v$ [1]. To construct $\widetilde{\mathcal{G}_v}$, we choose any spanning subtree Y of the quotient $\mathcal{G}_v / G_v$, and identify each connected component of the preimage of Y in $\mathcal{G}_v$ to a

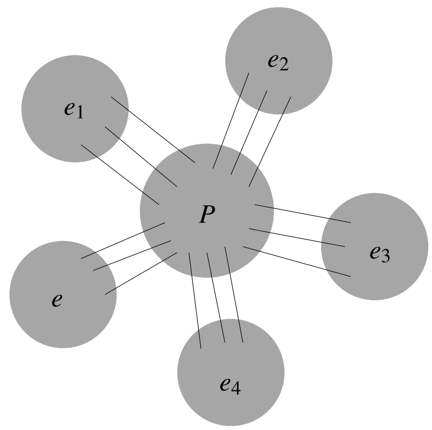

Figure 5: A vertex v

point. (Note that each such preimage contains exactly one vertex from each G_v–orbit.) Any $g \in G_v$ corresponds in $\widetilde{\mathcal{G}_v}$ to the image of the vertex $ga \in \mathcal{G}_v$; in particular, the vertices corresponding to the elements of the stabilizer G_e of e are the images in $\widetilde{\mathcal{G}_v}$ of the vertices of C_e, and those corresponding to the elements of any left coset gG_e of G_e in G to the images of the vertices of $gC_e = C_{ge}$. Furthermore, since $\mathcal{G}_v$ is 3–connected, the cyclic order of the vertices around ∂C_e coincides with that of their images under left multiplication by any $g \in G_v$ around $\partial(gC_e)$, and so the same is true in $\widetilde{\mathcal{G}_v}$. That is, $(\widetilde{\mathcal{G}_v}, G_e)$ is a planar pair.

Suppose there are two orbits of vertices under the G–action on T. Then G is the fundamental group of a graph of groups, where the graph may be taken to consist of e_0 and its two endpoints v and w in T. That is, G is the free product of G_v and G_w, amalgamating the subgroups $G_{e_0} = G_{\overline{e_0}}$. Furthermore, as above, $(\widetilde{\mathcal{G}_v}, G_{e_0})$ and $(\widetilde{\mathcal{G}_w}, G_{\overline{e_0}})$ are planar pairs, and it follows from the fact that $\mathcal{G}$ is three-connected that the identity map $1 : G_{e_0} \to G_{\overline{e_0}}$ respects the cyclic order of vertices—that is, $(G_v, G_{e_0}, G_w, G_{\overline{e_0}}, 1)$ is a planar 5–tuple, and $G = \mathrm{FPA}(G_v, G_{e_0}, G_w, G_{\overline{e_0}}, 1)$.

One the other hand, suppose there is just one orbit of vertices. Let $v_0 = [e_0]$. Then given any $e \in v_0$, there is an element $g \in G$ such that $gv_0 = \tau(e)$, and so $g^{-1}\overline{e} \in v_0$. Thus, v_0 contains translates of both e_0 and of $\overline{e_0}$. Since no vertex in any $\mathcal{G}_{g\overline{e_0}}$ lies in the same G_{v_0}–orbit as a, there is an element $t \in G$ and a vertex $v_1 \in \mathcal{G}_{t\overline{e_0}}$ such that a and v_1 belong to the same component of the preimage of Y, and so the image of v_1 in $\widetilde{\mathcal{G}_{v_0}}$ also represents $1 \in G_{v_0}$. Thus, any element of $G_{t\overline{e_0}}$ is represented in $\widetilde{\mathcal{G}_{v_0}}$ by the image of a vertex of $C_{t\overline{e_0}}$, and the elements of any left coset of $G_{t\overline{e_0}}$ are represented by the images of the vertices of the corresponding translate of $C_{t\overline{e_0}}$. Note that these images lie on the boundaries of disks in $\widetilde{\mathcal{G}_{v_0}}$ which are disjoint from those containing the cosets of G_{e_0}. Further, since $\mathcal{G}$ is three–connected, the homomorphism $t^* : G_{e_0} \to G_{t\overline{e_0}} = tG_{e_0}t^{-1}$ given by $t^*(g) = tgt^{-1}$ preserves the cyclic order of vertices. Thus, in this case, $(G_{v_0}, G_{e_0}, G_{t\overline{e_0}}, t^*)$ is a planar 4–tuple, and $G = \mathrm{HNN}(G_{v_0}, G_{e_0}, G_{t\overline{e_0}}, t^*)$.

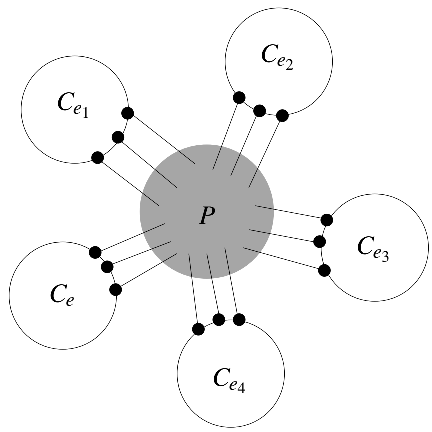

Figure 6: $\mathcal{G}_v$

Theorem 4.1 *Let G be a group, and suppose G has a planar Cayley graph with more than one end. Then G is either a free product with amalgamations* $\mathrm{FPA}(H,A,K,B,\phi)$ *where* (H,A,K,B,ϕ) *is a planar 5-tuple, or it is an HNN extension* $\mathrm{HNN}(H,A,B,\psi)$*, where* (H,A,B,ψ) *is a planar 4-tuple.*

PROOF. Let $\mathcal{G}$ be an embedding of a Cayley graph for G in $\mathbb{S}^2$. If $\mathcal{G}$ is either one– or two–separable, then the theorem follows from Theorems 4.1 and 4.4 of [3], and from the above discussion if $\mathcal{G}$ is three–connected. □

5 Accessibility

Theorem 5.1 *A finitely generated group with a planar Cayley graph is finitely presented.*

PROOF. Let G be such a group. Then G can be written as the fundamental group of a finite graph of groups whose edge groups all have order two or less, and whose vertex groups cannot be decomposed over $\mathbb{Z}_2$ [11] (that is, none of them has a graph of groups decomposition with an edge group of order one or two.) Each vertex group has a planar Cayley graph [1], and these are all three–connected [3, Theorem 2.1]. Therefore, by Whitney's theorem, the Cayley graphs of the vertex groups have planar embeddings in which the cyclic orders of the edge labels around any two vertices agree up to orientation. This implies that the vertex groups are finitely presented [10], which implies in turn that G is finitely presented. □

Possession of a planar Cayley graph is a Markov property, which means that there is no algorithm for deciding whether the group described by a given presentation has a planar Cayley graph, and so the best one can ask for is an effective enumeration of such groups.

A group is said to be *accessible* if it is the fundamental group of a finite graph of groups whose edge groups are all finite, and whose vertex groups are all one-ended. Finitely presented groups are accessible [7]. Suppose G has a planar Cayley graph. If G has more than one end, then it can be decomposed as a free product with amalgamations of some planar 5-tuple or an HNN extension of a planar 4-tuple, and in either case, the base groups also have planar Cayley graphs [1]. Thus, any of these which have more than one end can be further decomposed. The fact that G is accessible implies that this process must terminate after a finite number of steps [5]. Thus, any group with a planar Cayley graph can be "built" by a finite sequence of amalgamated free products and HNN extensions, beginning with the one–ended groups. This raises the question

Is there an effective enumeration of groups with planar Cayley graphs?

We cannot immediately answer this in the affirmative, due to the possibility that a group $\mathrm{FPA}(G,H,A,B,\phi)$ or $\mathrm{HNN}(G,A,B,\psi)$ may have a planar Cayley graph which is different in some essential way from the one obtained by the construction of section 2 from the planar Cayley graph(s) of its base group(s).

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