Surface Subgroups of Graph Groups

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Abstract

Let $\Gamma = (V, E)$ be a graph with vertex set V and edge set E. The *graph group* based on Γ, F_Γ, is the group generated by V, with defining relations $xy = yx$, one for each pair (x, y) of adjacent vertices in Γ. For $n \geq 3$, the *n-gon* is the graph with n vertices, $v_1, \ldots, v_n$, and n edges (v_i, v_{i+1}), indices modulo n. In this article we will show that if Γ has a full subgraph which is isomorphic to an n-gon, then the commutator subgroup of F_Γ, F'_Γ, has a subgroup which is isomorphic to the fundamental group of the orientable surface of genus $1 + (n - 4)2^{n-3}$.

So, in particular, the graph group of the pentagon contains a subgroup which is isomorphic to the group of the five-holed torus. As an application, we note that this implies that many artin groups contain surface groups, see [4]. We also use this result to study the commutator subgroups of certain graph groups, continuing the study of subgroups of graph groups begun in [2] and [6]. We show that F'_Γ is free if and only if Γ contains no full subgraph isomorphic to any n-gon with $n \geq 4$, which is an improvement on a theorem in [1]. We also show that if Γ contains no full squares, then F'_Γ is a graph group if and only if it is free; this shows that there exist graphs groups whose commutator subgroups are not graph groups.

1 Preliminaries

A graph is a pair (V, E), where V is the set of vertices and E is a set of unordered pairs of elements of V. So a graph is undirected with no loops or multiple edges. A graph $\Sigma = (W, D)$ is a subgraph of Γ if W is contained in V and D is contained in E. In this case there is a natural homomorphism $f : F_\Sigma \to F_\Gamma$ defined by setting $f(w) = w$. If D contains every pair vertices in W which are adjacent in Γ, then Σ is said to be a full subgraph. In this case it is clear that the natural homomorphism is one-to-one, and we shall regard F_Σ as a subgroup of F_Γ. If Γ has connected components $\Gamma_1, \Gamma_2, \dots, \Gamma_k$, then $F_\Gamma = F_{\Gamma_1} * F_{\Gamma_2} * \dots * F_{\Gamma_k}$. The *complement* of $\Gamma = (V, E)$, denoted by Γ^c, is the graph whose vertex set is also V, two vertices being adjacent in Γ^c if and only if they are not adjacent in Γ. If W is the vertex set of a connected component of Γ^c, then the full subgraph of Γ induced by W is called a *join component* of Γ. Note that two vertices which lie in different join components are adjacent in Γ.

Given $u \in F_\Gamma$, the *length of* u, $|u|$, is the length of a shortest word, w, in $V^{\pm 1}$ which represents it, and the *support* of u, $supp(u)$, is defined to be set of vertices $v \in V$ such that v or v^{-1} occurs in w. $Supp(u)$ is well defined, see [5]. u is *cyclically reduced* if u cannot be written $u = au'a^{-1}$ with $|u| = |u'| + 2$. Every element is conjugate to a cyclically reduced element. Let $x \in F_\Gamma$, x cyclically reduced. Let S_x be the full subgraph of $/Gamma$ induced by $supp(x)$, and let $A_1, \dots, A_n$ be the vertex sets of the join components of S_x. Then x can be factored uniquely as the product $x = x_1 \dots x_k$, where $supp(x_i) = A_i$. Let r_i be a generator for the largest cyclic subgroup of F_Γ containing x_i. r_i is uniquely defined up to sign and is called a *pure factor of* x. Let $link(x)$ denote the set of vertices v in $V - supp(x)$ such that v is adjacent to every vertex in $supp(x)$.

We will need the following theorem from [5] :

THEOREM 1 (CENTRALIZER THEOREM) *Let $x \in F_\Gamma$ and let r_i and $link(x)$ be defined as above. Then*

$$cent(x) = < r_1 > \otimes \cdots \otimes < r_n > \otimes < link(x) > .$$

2 Realization of the Commutator Subgroup

Let $\Gamma = (V, E)$ be a graph and let X_Γ denote the Cayley complex of the corresponding presentation of F_Γ; that is, X_Γ has one 0-cell, $*$, an oriented 1-cell for each of the n vertices in the graph Γ, and, for each edge (a, b) in E, an oriented two cell attached along the loop $aba^{-1}b^{-1}$. We have $\pi_1(X_\Gamma, *) = F_\Gamma$.

If Γ has n vertices, then X_Γ is a subcomplex of the n-fold cartesian product $(S^1)^n$, where the circle S^1 has one 0-cell and $(S^1)^n$ is the product complex. In particular, if K is the complete graph the vertex set V, then X_K is the entire 2-skeleton of $(S^1)^n$. Let U_K denote the universal cover of X_K. Since the fundamental group of any complex is carried by its 2- skeleton, it follows that U_K is the 2-skeleton of the cubical complex of R^n, i.e., the complex on R^n whose n- cells are integer translates of the unit cube I^n.

It is easy to see that $F_\Gamma/F'_\Gamma = Z^n$, the free abelian group of rank n. Thus F_K is the abelianization of F_Γ, and moreover, regarding Γ as a subgraph of K, the induced map $\alpha : F_\Gamma \to F_K$ is the natural projection of F_Γ onto its abelianization. The inclusion map $i : X_\Gamma \to X_K$ realizes α and we have that $F'_\Gamma = ker(i_*)$ is realized by U_Γ in the pullback diagram

$$\begin{array}{ccc} U_\Gamma & \longrightarrow & U_K \\ \downarrow & & \downarrow \\ X_\Gamma & \longrightarrow & X_K \end{array}$$

Hence X_Γ is the subcomplex of U_K obtained by deleting the lifts of all 2-cells in X_K which correspond to non-adjacent vertices of Γ.

3 Homotopies in X_Γ

Let w be a word on $V^{\pm 1}$ representing an element $[w] \in F_\Gamma$. Then, see [5] , w can be transformed into a word of shortest length representing $[w]$ via a finite sequence the following moves

M_1; delete the subwords $a^{-1}a$ or aa^{-1}, and

M$_1$; replace a subword $a^{\pm 1}b^{\pm 1}$ with $b^{\pm 1}a^{\pm 1}$, if $(a, b) \in E$.

In particular, if $[w] = 1$, then w can be transformed into the empty word by a finite sequence of these moves.

Let Z be a covering space of X_Γ, and let Z have the induced cell structure. Let $p : I \to Z$ be any cellular loop in Z which is path homotopic to the constant loop. Then, by the above, there is a sequence of path homotopies $p = p_1 \to p_2 \to \cdots \to p_k = *$ connecting p to the constant loop *, with each p_i cellular, such that the homotopy $p_i \to p_{i+1}$ is supported either by p_i, in the case of a move of type M$_1$, or, in the case of a move of type M$_2$, by $p_i \cup F$, where F is a face of Z which intersects p_i in at least two incident edges, (i.e. two edges which have a common endpoint. Let Y be a subcomplex of Z such that every face of Z which intersects Y in at least two incident edges also belongs to Y.

It follows that any loop in Y which is path homotopic to the constant path in Z is also path homotopic to the constant path in Y. This shows

PROPOSITION 1 *Let Z be a cover of the Cayley complex of F_Γ, and let Y be a subcomplex of Z with the property that any face of Z which contains at least two incident edges of Y is also a face of Y. Then the inclusion map $i : Y \to Z$ induces a monomorphism $i_* : \pi_1(Y) \to \pi_1(Z)$.*

We note that this yields a geometric proof of the fact that a full subgraph $\Omega \leq \Gamma$ induces an injection $F_\Omega \to F_\Gamma$.

4 Surface Subgroups of the n-gon Group

Let C_n denote the n-gon, and F_n its graph group. As in section 2, realize F'_n by a subcomplex U_n of the cubical lattice of R^n. Consider the subcomplex I^n of R^n. Since I^n is convex, every 2-cell of R^n which intersects I^n in two edges is also a 2-cell of I^n, and so $Y = U_n \cap I^n$ has the same property with respect to U_n. Thus, by proposition 1, it follows that $\pi_1(Y)$ is a subgroup of F'_n. We now show that Y is a 2-sided surface.

Every edge of I^n corresponds to a vertex v in C_n, and is incident to n-1 faces, one for each of the other n-1 vertices in C_n. Y contains exactly two of these faces; those corresponding to the two vertices in C_n which are adjacent to v. Thus Y is a connected 2-complex in which every edge is incident to exactly two faces, so Y is a surface. Y is 2-sided since it is a subcomplex of the 2-skeleton of I^n.

It remains only to compute the genus of Y. I^n has 2^n vertices, $n2^{n-1}$ edges, and $\binom{n}{2} 2^{n-2}$ faces, with 2^{n-2} faces for each of the $\binom{n}{2}$ pairs of vertices in C_n. Since only n of these pairs are adjacent in C_n, Y has only $n2^{n-2}$ faces, and the euler characteristic of Y is

$$\epsilon(Y) = 2^n - n2^{n-1} + n2^{n-2} = (4-n)2^{n-2},$$

and the genus of Y is $g(Y) = 1 - \frac{\epsilon(Y)}{2} = 1 + (n-4)2^{(n-3)}$. Thus we have shown

THEOREM 2 *Let F_n be the graph group of the n-gon graph. F_n' contains a subgroup isomorphic to the fundamental group of the orientable surface of genus* $1 + (n-4)2^{n-3}$.

5 Commutator Subgroups of Graph Groups

Let $\Gamma = (V, E)$ be a graph. A word w on $V^{\pm 1}$ represents an element of F_Γ' if and only if the exponent sum on each letter of V in w is 0. This is because the the exponent sum of each vertex in each relator in the graphic presentation for F_Γ is 0. We have noted that if Σ is a full subgraph of Γ, then the vertices of Σ generate a subgroup of F_Γ isomorphic to F_Σ. It follows that $F_\Sigma' = F_\Sigma \cap F_\Gamma'$.

A graph is said to be *triangulated* if it contains no full subgraph isomorphic to an n-gon with $n \geq 4$.

THEOREM 3 *If Γ is finite, then F_Γ' is free if and only if Γ is triangulated.*

PROOF: If Γ contains a full n-gon with $n \geq 4$, then F_Γ' contains a surface group of positive genus, and so cannot be free.

Conversely, suppose Γ is triangulated. If Γ is a complete graph then F_Γ' is trivial, so we suppose Γ is not complete. Then, see [3], Γ is the union of two full subgraphs X and Y such that $X \cap Y$ is either complete or empty. Comparing presentations, we see that F_Γ is the amalgamated free product of F_X with F_Y over $F_{X\cap Y}$. X and Y are both triangulated since they are full subgraphs of Γ. Therefore, by induction, $F_X \cap F_\Gamma' = F_X'$ and $F_Y \cap F_\Gamma' = F_Y'$, are both free, and $F_{X\cap Y} \cap F'_\Gamma = F'_{X\cap Y} = 1$ since $F_{X\cap Y}$ is abelian. Thus, since F'_Γ is normal in F_Γ, F'_Γ is free by the Kurosh subgroup theorem. □

The following theorem will allow us to conclude that the commutator subgroups of some graph groups are not graph groups.

THEOREM 4 *Let* Γ *contain no full squares, and let* $x \in F'_\Gamma$*,* x *nontrivial. Then the centralizer of* x *in* F_Γ*,* $cent(x)$*, is free abelian, and the centralizer of* x *in* F'_Γ*,* $cent(F'_\Gamma; x)$*, is cyclic.*

PROOF: Recall from section 1 the notation of the centralizer theorem. We may suppose that x is cyclically reduced. Since $x \neq 1$ and $x \in F'_\Gamma \cap F_{supp(x)} = F'_{supp(x)}$, $supp(x)$ must contain two non-adjacent vertices, say a and b. Therefore $link(x)$ induces a complete subgraph of Γ, since if c and d are non-adjacent vertices of $link(x)$, then a, b, c and d form a full square. See figure 1. By the same argument, the supports of all but one of the pure

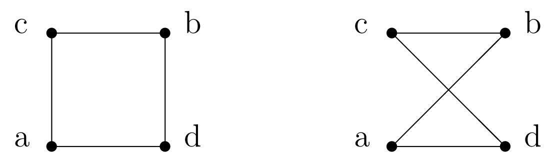

Figure 1: Two views of the square

factors of x induce complete subgraphs of Γ. Since each pure factor lies in F'_Γ, it follows that x has only one pure factor r. Therefore, by the Centralizer Theorem, $cent(x) =< r > \otimes < link(x) >$, which is free abelian. Also,

$$cent(F'_\Gamma; x) = cent(x) \cap F'_\Gamma =< r > x < link(x) >' =< r >,$$

since $< link(x) >$ is free abelian. □

COROLLARY 1 *If* Γ *contains no full squares and* $F_{\Gamma'}$ *is a graph group, then* F'_Γ *is free.*

PROOF: If F'_Γ is a graph group with graph Σ, then, since the valence of any vertex σ in Σ is equal to the rank of the abelianization of $cent(\sigma)$ in F_Σ. By theorem 3, $cent(\sigma)$ is cyclic, so Σ is discrete and F'_Γ is free. □

Corollary 2 *If Γ contains no full squares but does contain a full n-gon for $n > 4$, then F'_Γ is not a graph group.*

Proof: Let C be a full subgraph which is an n-gon, $n > 4$. Then $F'_C = F_C \cap F'_\Gamma$ is a subgroup of F'_Γ, and so, by Theorem 1, F'_Γ contains a surface group of positive genus. No surface group is free, hence F'_Γ is not free and, by Corollary 1, not a graph group. □

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