

Tree groups and the 4-string pure braid group

Carl Droms*

James Madison University, Harrisonburg, VA 22807, USA

Jacques Lewin

Syracuse University, Syracuse, NY 13244-1154, USA

Herman Servatius

College of the Holy Cross, Worcester, MA 01610, USA

Communicated by K.W. Gruenberg

Received 22 December 1988

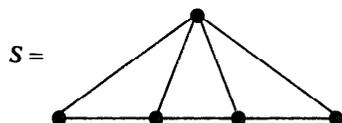
Revised 23 October 1989

Abstract

Droms, C., J. Lewin and H. Servatius, Tree groups and the 4-string pure braid group, Journal of Pure and Applied Algebra 70 (1991) 251–261.

Given a graph Γ , undirected, with no loops or multiple edges, we define the *graph group* on Γ , F_Γ , as the group generated by the vertices of Γ , with one relation $xy = yx$ for each pair x and y of adjacent vertices of Γ .

In this paper we will show that the unpermuted braid group on four strings is an HNN-extension of the graph group F_S , where



The form of the extension will resolve a conjecture of Tits for the 4-string braid group. We will conclude, by analyzing the subgroup structure of graph groups in the case of trees, that for any tree T on a countable vertex set, F_T is a subgroup of the 4-string braid group.

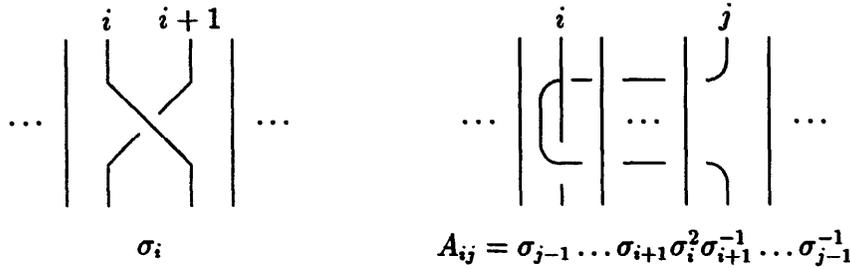
We will also show that this uncountable collection of subgroups of the 4-string braid group is linear, that is, each subgroup embeds in $GL(3, \mathbb{R})$, as well as embedding in $\text{Aut}(F)$, where F is the free group of rank 2.

* This author was supported by a summer research grant from James Madison University.

1. Tree groups and braid groups

In this section we are concerned only with the four string braid group, B_4 ,

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_1\sigma_3 = \sigma_3\sigma_1 \rangle.$$



See [3] for background on the braid group. The main result here is that the graph group on $\bullet - \bullet - \bullet - \bullet$ embeds naturally into B_4 .

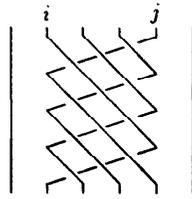
There is a map $B_4 \rightarrow S_4$, into the symmetric group which associates to each braid the permutation of the four strings it effects. The kernel of this map is $P = P_4$, the *pure* or *unpermuted* braid group. P has a presentation, see [10, p. 174], as follows:

Generators: $A_{12}, A_{23}, A_{34}, A_{13}, A_{24}, A_{14}$.

Relations:

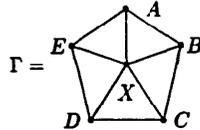
$$\begin{aligned} A_{12}A_{34} &= A_{34}A_{12}, \\ A_{14}A_{23} &= A_{23}A_{14}, \\ A_{24}A_{13}A_{24}^{-1} &= A_{14}^{-1}A_{12}^{-1}A_{14}A_{12}A_{13}A_{12}^{-1}A_{14}^{-1}A_{12}A_{14}, \\ A_{23}A_{12}A_{23}^{-1} &= A_{13}^{-1}A_{12}A_{13}, \\ A_{24}A_{12}A_{24}^{-1} &= A_{14}^{-1}A_{12}A_{14}, \\ A_{34}A_{13}A_{34}^{-1} &= A_{14}^{-1}A_{13}A_{14}, \\ A_{34}A_{23}A_{34}^{-1} &= A_{24}^{-1}A_{23}A_{24}, \\ A_{23}A_{13}A_{23}^{-1} &= A_{13}^{-1}A_{12}^{-1}A_{13}A_{12}A_{13}, \\ A_{24}A_{14}A_{24}^{-1} &= A_{14}^{-1}A_{12}^{-1}A_{14}A_{12}A_{14}, \\ A_{34}A_{14}A_{34}^{-1} &= A_{14}^{-1}A_{13}^{-1}A_{14}A_{13}A_{14}, \\ A_{34}A_{24}A_{34}^{-1} &= A_{24}^{-1}A_{23}^{-1}A_{24}A_{23}A_{24}. \end{aligned}$$

The subgroup of P consisting of all braids which are concentrated on strings i through j has [9] an infinite cyclic center generated by $D_{ij} = (\sigma_i\sigma_{i+1}\cdots\sigma_{j-1})^{j-i+1}$. The element D_{ij} geometrically represents a full twist of the i th through j th strings through a full 360 degrees.



The element D_{ij} .

There is thus a map f from the graph group on the graph



into P , $f: F_\Gamma \rightarrow P$, defined by $f(A) = D_{12}$, $f(B) = D_{13}$, $f(C) = D_{23}$, $f(D) = D_{24}$, $f(E) = D_{34}$, and $f(X) = D_{14}$. The relations of F_Γ are preserved under f , with $D_{12}D_{34} = D_{34}D_{12}$ since D_{12} and D_{34} are concentrated on disjoint sets of strings. f is also surjective since we have the equations

$$A_{12} = D_{12}, \quad A_{13} = D_{12}^{-1}D_{13}D_{23}^{-1}, \quad A_{14} = D_{14}D_{13}^{-1}D_{23}D_{24}^{-1},$$

$$A_{23} = D_{23}, \quad A_{24} = D_{23}^{-1}D_{24}D_{34}^{-1}, \quad A_{34} = D_{34}.$$

It is straightforward to verify that the kernel of f is the normal closure of the element $AB^{-1}CD^{-1}EA^{-1}BC^{-1}DE^{-1}$. Setting Γ' equal to the outer pentagon of Γ , it follows that P is isomorphic to $\langle X \rangle \oplus G$, where G is the group obtained from $F_{\Gamma'}$ by adjoining the single relation $AB^{-1}CD^{-1}E = ED^{-1}CB^{-1}A$.

Any four consecutive vertices of the pentagon generate a three line group, say, $\{A, B, C, D\}$. Setting $\Omega = \overset{A}{\bullet} - \overset{B}{\bullet} - \overset{C}{\bullet} - \overset{D}{\bullet}$ we have homomorphisms $F_\Omega \rightarrow F_{\Gamma'} \rightarrow G$, where the first map is simply the inclusion, and we want to show this composite is injective. This follows from the following:

Theorem 1. G is an HNN-extension of F_Ω .

Proof. Consider the set $\{A, D, AB^{-1}CD^{-1}\}$. These three elements generate a free subgroup of rank three in F_Ω . This is because the so-called ‘edge group’, $\langle AB^{-1}, B^{-1}C, C^{-1}D \rangle$ is free [6], and the automorphism of F_Ω induced by

$$A \rightarrow AB; \quad B \rightarrow B; \quad C \rightarrow C; \quad D \rightarrow CD$$

carries this subgroup onto $\langle A, B^{-1}C, D \rangle$ which equals $\langle A, AB^{-1}CD^{-1}, D \rangle$ by Nielsen transformations. Similarly $\langle A, D^{-1}CB^{-1}A, D \rangle$ is free of rank 3, and so there is an isomorphism

$$f: \langle A, D, D^{-1}CB^{-1}A \rangle \rightarrow \langle A, AB^{-1}CD^{-1}, D \rangle$$

given by $f(A) = A$, $f(D) = D$, and $f(D^{-1}CB^{-1}A) = f(AB^{-1}CD^{-1})$. Let H be the HNN-extension

$$\langle F_\Omega, E \mid g^E = f(g), g \in \langle A, D, AB^{-1}CD^{-1} \rangle \rangle.$$

By examining presentations, we see that H is isomorphic to G . \square

Note that the other four three-line groups also inject by symmetry.

Corollary 1. F_Ω embeds into P . \square

Corollary 1, together with Corollary 6 give the following:

Corollary 2. The pure 4-string braid group has uncountably many non-isomorphic subgroups. \square

The following corollary answers in the affirmative, in the case of the 4-string braid group, a conjecture due to Tits [1].

Corollary 3. The subgroup of B_4 generated by D_{12} , D_{23} and D_{34} has the presentation $\langle D_{12}, D_{23}, D_{34} \mid [D_{12}, D_{34}] \rangle$.

Proof. The subgroup of B_4 generated by D_{12} , D_{13} , D_{23} and D_{34} has the presentation

$$\langle D_{12}, D_{13}, D_{23}, D_{34} \mid [D_{12}, D_{13}], [D_{13}, D_{23}], [D_{12}, D_{34}] \rangle.$$

and we have already noted [8], that every subset of the vertices of a graph group generates a subgroup which is itself a graph group on their induced subgraph. \square

Corollary 4. Let P denote the pure 4-string braid group, $P = \langle X \rangle \oplus G$ as above. The presentation for G is Cohen-Lyndon aspherical.

Proof. This follows from Theorems 4.3 and 3.7 of [4]. \square

2. Subgroups of graph groups

If Γ is finite, then [8] every finitely-generated subgroup of F_Γ is itself a graph group if and only if no full subgraph of Γ is isomorphic to either

$$\Omega = \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \quad \text{or} \quad C_4 = \begin{array}{cc} \bullet & \bullet \\ | & | \\ \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array}.$$

The graph group on C_4 is the direct product of two free groups of rank 2, whose subgroup structure was examined in [2]. In this section, we consider the related question: If Γ is a graph, for which graphs Φ does F_Γ have a subgroup isomorphic to F_Φ ? We will answer this question in the case that Γ is a tree.

If F_Φ is isomorphic to a subgroup of F_Γ , then we say F_Φ is a *graph subgroup* of F_Γ . In particular, if Φ is a full subgraph of Γ , that is, every pair of vertices in Φ which are adjacent in Γ are also adjacent in Φ , then the subgroup of F_Γ generated by the vertices of Φ is isomorphic to F_Φ . In this case we call F_Φ a *subgraph subgroup* of F_Γ . Henceforth, we will use the term ‘subgraph’ to mean ‘full subgraph’.

We will show that, for any subgraph Φ of Γ , the normal closure of the subgraph subgroup F_Φ in F_Γ is itself a graph group. Indeed, one can describe quite simply a set of ‘graphic’ generators and relations for this subgroup. We denote by Φ' the subgraph of Γ generated by the vertices of Γ which are not in Φ . In this section we will prove the following:

Theorem 2. *The normal closure of F_Φ in F_Γ has a presentation with generators g' , one for each distinct nontrivial conjugate $g = a^{-1}\mu a \in F_\Gamma$, where $a \in F_\Phi$ and μ is a vertex of Φ , and one relator $[g', h']$ for each such pair g, h that commutes in F_Γ .*

Proof. For any graph Λ , let S_Λ be the standard complex for F_Λ , so that Λ has one 0-cell, 1, the base point, a 1-cell v for each vertex v of Λ , and a 2-cell $v_1 - v_2$ for each edge of Λ . The 2-cell $v_1 - v_2$ is attached to the path $v_1 v_2 v_1^{-1} v_2^{-1}$. By construction, $\pi_1(S_\Lambda) = \pi_1(S_\Lambda, 1) = F_\Lambda$.

In the notation of the theorem, there is a retraction $f: S_\Gamma \rightarrow S_{\Phi'}$, which collapses the 1-cells corresponding to vertices of Φ to 1. If $v_1 - v_2$ is an edge of Φ , $v_1 - v_2$ is collapsed to 1. If $v_1 - v_2$ has $v_1 \in \Phi$, $v_2 \in \Phi'$, then the cell $v_1 - v_2$ is collapsed to the 1-cell v_2 of Φ' . The kernel $K = \ker f_*$ of the induced map $f_*: F_\Gamma \rightarrow F_{\Phi'}$ is clearly the normal closure of the subgroup F_Φ in F_Γ .

The universal cover $U_{\Phi'}$ of $S_{\Phi'}$ has an induced cell structure as a 2-dimensional complex and the 1-skeleton is the Cayley graph of $F_{\Phi'}$. Consider the pullback diagram

$$\begin{array}{ccc} Y & \xrightarrow{f'} & U_{\Phi'} \\ p' \downarrow & & \downarrow p \\ S_\Gamma & \xrightarrow{f} & S_{\Phi'} \end{array}$$

where Y is the subspace of $S_\Gamma \times U_{\Phi'}$ consisting of all points (x, y) with $f(x) = p(y)$. Since the pullback of a covering map is a covering map, the induced map p'_* in

$$\begin{array}{ccc} \pi_1(Y) & \longrightarrow & 1 \\ p'_* \downarrow & & \downarrow \\ F_\Gamma & \xrightarrow{f_*} & F_{\Phi'} \end{array}$$

is injective and maps into K . The loops $v_1 v v_1^{-1}$ with $v \in \Phi$, $v_1 \in \Phi'$ of $S_{\Phi'}$ are easily seen to lift to loops of Y , so p'_* gives an isomorphism $\pi_1(Y) \cong K$.

It is straightforward to give a convenient decomposition of Y as a 2-complex. First, by considering $S_{\phi'}$ as a subcomplex of S_{Γ} , and identifying $U_{\phi'}$ with the graph of p , we see that we can take $U_{\phi'}$ to be a subcomplex of Y .

We take $F_{\phi'}$, the 0-skeleton of $U_{\phi'}$, to be the 0-skeleton of Y .

The 1-cells of $U_{\phi'}$ are edges $g \text{---} gv$, $v \in \Phi$, $g \in F_{\phi}$. The loops

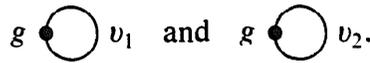


$v \in \Phi$ of S_{Γ} pull back to loops

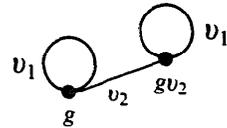


$g \in F_{\Gamma}$.

Aside from the 2-cells of $U_{\phi'}$, we have 2-cells which are pullbacks of 2-cells of $S - \Gamma$. If $v_1 \text{---} v_2$ is a 2-cell with both $v_1, v_2 \in \Phi$, it pulls back to tori in Y , one for each $g \in F_{\phi'}$, whose meridians and longitudes are



If $v_1 \in \Phi$ and $v_2 \in \Phi'$, then $v_1 \text{---} v_2$ pulls back to a cylinder attached along:



It is easily checked that every point of Y lies on one of the cells described above.

We now consider the quotient complex \bar{Y} which is obtained from Y by identifying the subcomplex $U_{\phi'}$ to a point. Since $U_{\phi'}$ is connected and simply connected, the exact homotopy sequence of a pair

$$\cdots \rightarrow \pi_1(A, 1) \rightarrow \pi_1(Y, 1) \rightarrow \pi_1(Y, A) \rightarrow \pi_0(A, 1) \rightarrow \cdots$$

implies that the quotient map $Y \rightarrow \bar{Y}$ induces an isomorphism $\pi_1(Y, 1) \rightarrow \pi_1(\bar{Y}, \bar{1}) = \pi_1(Y, A)$. \bar{Y} is a 2-complex with one 0-cell, the class of 1, so it is a standard complex, and a presentation for $\pi_1(Y, 1) \cong \pi_1(\bar{Y}, \bar{1})$ can be read off from the induced cell structure of \bar{Y} . If we write v^g for the 1-cells



then we have as generators for K the set $\{v^g \mid v \in \Phi, g \in F_{\phi'}\}$. The relations are:

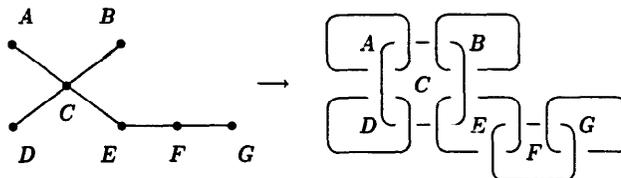
- (1) $v^g = v^{g'w}$ if $v \text{---} w$ is an edge of Γ with $v \in \Phi$ and $w \in \Phi'$,
- (2) $[v^g, w^g] = 1$ if $v \text{---} w$ is an edge of Φ .

The centralizer $C_{\phi'}(v)$ of v in $F_{\phi'}$ is generated by the elements of ϕ' adjacent to v , as is easily seen from the normal form of elements in F_{Γ} (or see [11]). The rela-

tions in (1) can then be eliminated if we choose for generators of K elements v^g in 1:1 correspondence with the cosets of C_ϕ .

Since the element v^g is in fact the conjugate gv^g^{-1} , the theorem is proved. \square

We now consider trees. It has been shown [6] that if T is a finite tree, then F_T is a three-manifold group. It can be shown, in fact, that for any locally finite tree T , F_T is the group of a link in \mathbb{R}^3 . The construction of the link in the case of a finite tree is illustrated in the diagram below:



So to each vertex in the tree there corresponds one unknotted circle, linked once to each of its neighbors as shown. A van Kampen argument shows that the fundamental group of the complement of this link is F_T , and by [7] it follows that two distinct trees give two non-equivalent links. Nevertheless, any two finite trees with the same number of vertices yield links with identical Jones polynomials, as can easily be verified.

We will now characterize the graph subgroups of tree groups. A tree T is called a star if it has one vertex which is adjacent to all the others. Note that this is the case if and only if T has no subgraph isomorphic to L_3 , and that if T is a star, then F_T is the direct product of an infinite cyclic group and a free group.

Proposition 1. *If T is a star, then every subgroup of F_T is either free or isomorphic to F_S for some star S .*

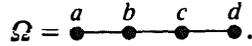
Proof. Since T is a star, F_T is isomorphic to $\mathbb{Z} \oplus F$, where F is free. Thus, if $H < F_T$, then H is a semidirect product of $H \cap \mathbb{Z}$ and the projection of H onto the factor F , and since \mathbb{Z} lies in the center of F_T , the product is direct. Thus, H is isomorphic to $(H \cap \mathbb{Z}) \oplus K$, where K is free. Since $H \cap \mathbb{Z}$ is either trivial or infinite cyclic, the result follows. \square

Theorem 3. *If T is a tree which is not a star, then every graph subgroup of T is isomorphic to F_ϕ for some countable forest ϕ . Conversely, if ϕ is any countable forest, then F_T has a subgroup isomorphic to F_ϕ .*

Proof. Let F_ϕ be a subgroup of F_T . By [6], F_T is coherent. Thus F_ϕ is coherent, which, by [6], implies that every circuit of ϕ has a chord. To show that ϕ must be a forest, it will suffice to show that ϕ may contain no triangles. Suppose ϕ contains a triangle Σ . Then there is a finite subgraph T' of T such that $F_\Sigma < F_{T'}$. But T' is a

forest, so by [5], $cd(F_T) \leq 2$, which is a contradiction, since F_Σ is free abelian of rank 3.

It suffices to prove the second statement for the graph



It follows from Theorem 2 that the normal closure of the elements b, c in F_Ω is generated by the distinct conjugates of b and c , with defining relators those commutators among pairs of these conjugates which are trivial in F_Ω . The corresponding graph Φ is a forest, by the above, and in fact each vertex of Φ has infinitely many neighbors; consider the element $g^{-1}bg$, where $g \in \text{gp}\langle a, d \rangle$. The elements $(g^{-1}a^{-n})c(a^n g)$ are distinct for distinct $n \in \mathbb{Z}$, and each commutes with $g^{-1}bg$, so the vertex in Φ corresponding to $g^{-1}bg$ has infinitely many neighbors.

Now, it is clear that if Σ is any countable forest, then the graph Φ contains an isomorphic copy of Σ as a full subgraph, and hence F_Ω contains a subgroup isomorphic to F_Σ . \square

Corollary 5. *The graph subgroups of F_Ω consist of the graph groups on countable forests.* \square

The following corollary is analogous to a result in [2] for the direct product of two free groups of rank 2.

Corollary 6. *If the graph Γ contains a full subgraph isomorphic to Ω , then F_Γ has uncountably many non-isomorphic subgroups.*

Proof. This follows immediately from the fact that if the graphs Γ and Φ are not isomorphic, then neither are the groups F_Γ and F_Φ [7], and that there are uncountably many countable trees. \square

3. Representations of tree groups¹

The question of the linearity of the braid groups has a long history. The purpose of this section is to prove the following:

Theorem 4. *The group F_Ω has a faithful, 3×3 , real representation, $\Omega =$*

$\bullet \xrightarrow{A} \bullet \xrightarrow{B} \bullet \xrightarrow{C} \bullet \xrightarrow{D}$

Proof. Let g be the group with presentation

$$\langle A, B, T \mid [A, B] = 1, [B, B^T] = 1, T^2 = 1 \rangle.$$

¹The authors acknowledge with thanks the suggestions of A. Schofield.

Then, applying the Reidemeister-Schrier process, the normal closure of the elements $A, B \in G$ is isomorphic to F_Ω under the correspondence

$$A \rightarrow A, \quad B \rightarrow B, \quad C \rightarrow B^T, \quad D \rightarrow A^T.$$

Thus, it will suffice to show that G has such a representation. Let

$$G_1 = \langle A, B \mid [A, B] = 1 \rangle,$$

$$G_2 = \langle B, T \mid [B, B^T] = 1, T^2 = 1 \rangle.$$

Then G is isomorphic to the amalgamated free product $G_1 \star_{\langle B \rangle} G_2$.

We will first write faithful representations of G_1 and G_2 :

G_1 : G_1 is faithfully represented by the group M_1 generated by the matrices A and B , where

$$A = \begin{pmatrix} y & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

x, y and z are algebraically independent transcendentals.

G_2 : The normal closure in G_2 of the element B is a free abelian group generated by B and B^T . Thus, G_2 is the split extension of this subgroup by the group of order two generated by T . Let T and B be the matrices

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B^T = \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x \end{pmatrix}.$$

Then B^T and B indeed generate a free abelian group of rank 2, so the group M_2 generated by B and T is isomorphic to G_2 by the 5-lemma.

Note that these representations are over the field $\mathbb{Q}(x, y, z)$.

Let U be the space spanned by the column vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and W that spanned by

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$A^n B^m = \begin{pmatrix} y^n x^m & 0 & 0 \\ 0 & z^n x^m & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the only such matrices which act as scalars on $U \bmod W$ are the powers of B (i.e., those with $n=0$).

In M_2 , every element can be written in one of the two forms $B^n C^m$ or $TB^n C^m$, where $C = B^T$.

$$B^n C^m = \begin{pmatrix} x^{n+m} & 0 & 0 \\ 0 & x^n & 0 \\ 0 & 0 & x^m \end{pmatrix} \quad \text{and} \quad TB^n C^m = \begin{pmatrix} x^{n+m} & 0 & 0 \\ 0 & 0 & x^m \\ 0 & x^n & 0 \end{pmatrix}$$

and so the only matrices in M_2 that act as scalars on $U \bmod W$ are again the powers of B . Thus, by Wehrfritz's Theorem 3 in [12], the group $M_1 \star_{\langle B \rangle} M_2$, which is isomorphic to G , has a faithful 3-dimensional representation over a purely transcendental extension of the field $\mathbb{Q}(x, y, z)$, and hence over \mathbb{R} . \square

We define now the group $G = \langle a, b \mid [a^b, a] \rangle$. G contains the infinite line group as the normal closure of a ,

$$\dots \text{---} b^2 a b^{-2} \text{---} b a b^{-1} \text{---} a \text{---} b^{-1} a b \text{---} b^{-2} a b^2 \text{---} \dots,$$

again by the Reidemeister-Schrier process, and hence G contains a copy of the three-line group and hence every forest group. It is also true that G has a faithful representation in 4×4 matrices over \mathbb{R} as follows. Let X, T and T' be defined by

$$X = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}, \quad T = \begin{pmatrix} t_1 & t_2 \\ t_4 & t_3 \end{pmatrix}, \quad T' = \begin{pmatrix} t'_1 & t'_2 \\ t'_4 & t'_3 \end{pmatrix},$$

where the nonzero entries are algebraically independent transcendentals.

Let π be the 4×4 matrix corresponding to the transposition (2 3).

Then a is represented by $\begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$ and b by $\pi \begin{pmatrix} T & 0 \\ 0 & T' \end{pmatrix}$. Thus every line group has a 4×4 representation in which the generators are conjugates.

We may also represent tree groups in the automorphism group of a free group.

Theorem 5. *F_Ω , and hence every countable forest group, is contained in the group of automorphisms of a free group of rank 2.* \square

The group $G = \langle a, b \mid [a^b, a] \rangle$, defined above, also has the presentation

$$G = \langle x, y, t \mid x^t = xy, y^t = y \rangle,$$

under the correspondence $t = a$, $x = b$, and $y = [b, a]$. Note that the subgroup generated by x and y is normal, free of rank 2, and the quotient is infinite cyclic, in fact, it is the normal closure of b . Thus, G fits into a split exact sequence

$$1 \rightarrow F_2 \rightarrow G \rightarrow Z \rightarrow 1,$$

where $F_2 = \text{gp}\langle x, y \rangle$.

Now, as usual, to this extension corresponds an automorphism s of F_2 ; in this case, it is the automorphism

$$s(x) = xy, \quad s(y) = y.$$

It is not hard to see that G is actually the subgroup of $\text{Aut}(F_2)$ generated by the inner automorphisms and the automorphism s . Thus, every forest group is contained in $\text{Aut}(F_2)$.

References

- [1] K. Appel and P. Schupp, Artin groups and infinite coxeter groups, *Invent. Math.* 72 (1983) 201–220.
- [2] G. Baumslag and J. Roseblade, Subgroups of direct products of free groups, *J. London Math. Soc.* (2) 30 (1984) 44–52.
- [3] J. Birman, *Braids Links and Mapping Class Groups* (Princeton University Press, Princeton, NJ, 1974).
- [4] I.M. Chiswell, D.J. Collins and J. Huebschmann, Aspherical group presentations, *Math. Z.* 178 (1981) 1–36.
- [5] W. Dicks, An exact sequence for rings of polynomials in partly commuting indeterminates, *J. Pure Appl. Algebra* 22 (1981) 215–228.
- [6] C. Droms, Graph groups, coherence, and three-manifolds, *J. Algebra* 106 (2) (1987) 484–489.
- [7] C. Droms, Isomorphisms of graph groups, *Proc. Amer. Math. Soc.* 100 (3) (1987) 407–408.
- [8] C. Droms, Subgroups of graph groups, *J. Algebra* 110 (1987) 519–522.
- [9] F.A. Garside, The braid group and other groups, *Quart. J. Math. Oxford* (2) 20 (1969) 235–254.
- [10] W. Magnus, A. Karrass and D. Solitar, *Combinatorial Group Theory* (Dover, New York, 2nd rev. ed., 1976).
- [11] H. Servatius, Automorphisms of graph groups, *J. Algebra* 126 (1) (1989) 34–60.
- [12] B.A.F. Wehrfritz, Generalized free products of linear groups, *Proc. London Math. Soc.* (3) 27 (1973) 402–424.