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The Tits Conjecture and the Five String Braid Group

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It has been conjectured by J. Tits [1] that all relations satisfied by the squares of the generators in any Artin group are consequences of the relations of commutativity satisfied by the generators. In this article, we show that this conjecture is true for the 5-string braid group.

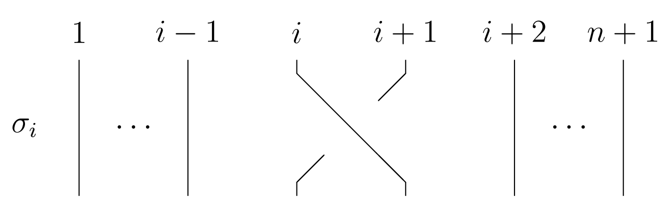
Let $\Gamma = (V, E, \mu)$ be a weighted graph with weight function μ assigning to each edge e of Γ an integer $\mu(e) \geq 2$. The Artin group A_Γ is the group generated by the vertices of Γ with, for each edge $e = x, y$ of Γ, the relation $xyxy \cdots = yxyx \cdots$, where each side consists of $\mu(e)$ factors. We call Γ the Artin graph of the group A_Γ.

J. Tits [1] has conjectured that all relations satisfied by the squares of the given generators of A_Γ are consequences of the relations $x^2y^2 = y^2x^2$, where $e = x, y$ is an edge of Γ, and $\mu(e) = 2$. That is, that the subgroup of A_Γ generated by these elements is isomorphic to the group A_{Γ_2} where Γ_2 is the graph obtained by deleting from Γ all edges e of weight $\mu(e) \neq 2$. K. Appel and P. Schupp, [1], have shown that the conjecture is true if all edges have weight greater than 3. S. Pride [4] has shown the conjecture is true for all Artin groups whose graphs contain no triangles.

As special cases of the Artin groups we have the classical braid groups. The n-string braid group, B_n, is generated by elements σ_i, $1 \leq i < n$, with two kinds of defining relations:

$$\begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for } |i-j| > 1, \text{and} \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for } 1 \leq i \leq n-2. \end{array}$$

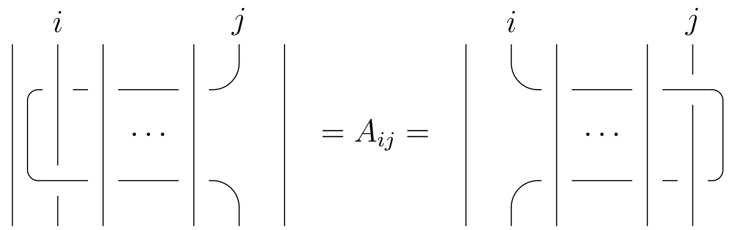
The generator σ_i denotes the braid:



Note that the Artin graph of B_n is K_{n-1}, the complete graph on $n-1$ vertices, with all edges labelled either 2 or 3, and the edges of weight three constituting the hamiltonian path $\{\sigma_1, \cdots, \sigma_{n-1}\}$. It is straightforward to verify that the Tits conjecture is true for B_2 and B_3 and we have previously shown [3] that it holds for B_4. In this paper, we will show that it holds as well for B_5.

We shall also be interested in Artin groups A_Γ such that $\mu(e) = 2$ for each edge of Γ. We call such an Artin group a *graph group* and write $A_\Gamma = F_\Gamma$. Thus, in the presentation of a graph group, each relation expresses the fact that some two generators commute.

The n-string *pure braid group* is the kernel P_n of the homomorphism $\pi : B_n \to S_n$, where S_n denotes the symmetric group on n letters, which sends each braid to the permutation of the strings that it effects (see [2] for details). P_n is generated by the braids A_{ij}:



In particular, $A_{i,i+1} = \sigma_i^2$. The center of P_n is infinite cylic, generated by the element $D_n = (\sigma_1 \cdots \sigma_{n-1})^n$. The generator of the center of P_4 is $D_4 = (\sigma_1\sigma_2\sigma_3)^4$ and is pictured in figure 1.

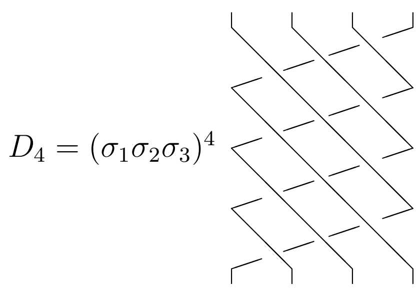
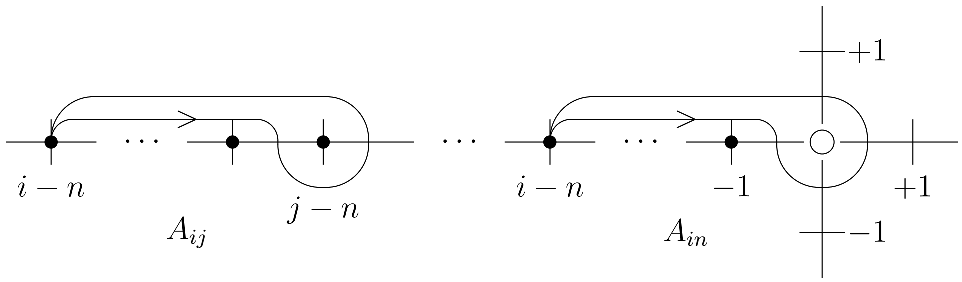


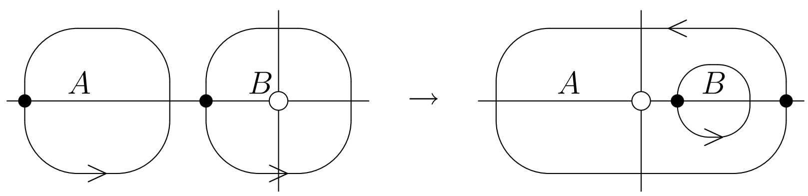
Figure 1: The element $D_4 = (\sigma_1\sigma_2\sigma_3)^4$

The pure braid group, P_n, can also be described as the fundamental group of the space $M^n = \mathbf{C}^n - X$, where $\mathbf{C}$ is the complex plane and X is the *grand diagonal* of $\mathbf{C}^n$, that is, the set of all n-tuples of complex numbers which have the same value in at least two coordinates. So X is a union of complex hyperplanes. Pick as a base point $\beta = (-(n-1), \cdots, -1, 0)$, so every element of P_n is represented by a loop $x : ([0,1], 0, 1) \to (M^n, \beta)$. If we map M^n onto itself by $(z_1, \cdots, z_n) \to (z_1, \cdots, z_n) - \alpha(z_n, \cdots, z_n)$, then for $0 \le \alpha \le 1$ there is a one parameter family of base-point preserving maps connecting the identity with the retraction $(z_1, \cdots, z_n) \to (z_1, \cdots, z_n) - (z_n, \cdots, z_n)$. The image of the retraction may be regarded as the space Y_n of $n-1$-tuples of distinct non-zero complex numbers with base point $\beta' = (-(n-1), -(n-2), \cdots, -1)$. The retraction map, by the above homotopy, induces an isomorphism of the fundamental groups of Y^n and M^n under which the generators A_{ij}, correspond to the loops pictured below.

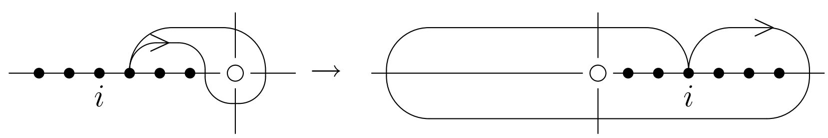
Now, map $C - 0$ onto itself by $\pi : z \to -1/z$. This defines a homeomorphism of Y^n onto itself sending the base point to $\{(n-1)^{-1}, (n-2)^{-1}, \cdots, 2^{-1}, 1\}$. Notice that the order of the $n-1$ points on the real axis is preserved.

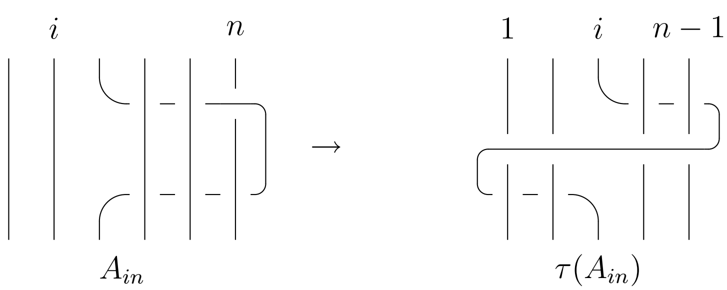


The effect of π on the key elements of P_n can be determined from the following diagram which illustrates the effect of $z \to -1/z$ on the the oriented circles A and B.



So, noting that the generators A_{ij} are made up of such arcs, we may draw their images, and, ignoring the puncture in the image space, we obtain a map $\tau : P_n \to P_{n-1}$ defined by sending A_{ij}, $j < n$ onto itself with $\tau(A_{in})$ defined via the diagram below.





$$
\begin{aligned}
\tau(A_{ij}) &= (A^{-1}_{i,n-1} \cdots A^{-1}_{i,i+1})(A^{-1}_{i-1,i} \cdots A^{-1}_{1,i}) \\
&= (\sigma_i^{-1} \cdots \sigma^{-1}_{n-2})(\sigma^{-1}_{n-2} \cdots \sigma_1^{-1})(\sigma_1^{-1} \cdots \sigma^{-1}_{i-1})
\end{aligned}
$$

On P_5 we have

$$\begin{aligned}
A_{12} = \sigma_1^2 &\longrightarrow \sigma_1^2, \\
A_{23} = \sigma_2^2 &\longrightarrow \sigma_2^2, \\
A_{34} = \sigma_3^2 &\longrightarrow \sigma_3^2, \text{ and} \\
A_{45} = \sigma_4^2 &\longrightarrow A_{34}^{-1} A_{24}^{-1} A_{14}^{-1} = D_4^{-1}(\sigma_1\sigma_2)^3.
\end{aligned}$$

It is a consequence of Theorem 3 in [3] that the subgroup of P_4 generated by the elements σ_1^2, σ_2^2, σ_3^2 and $D_4^{-1}(\sigma_1\sigma_2)^3$ is in fact a graph group on the graph

$$\sigma_3^2 \text{—} \sigma_1^2 \text{—} D^{-1}(\sigma_1\sigma_2)^3 \text{—} \sigma_2^2$$

It follows that the subgroup of P_5 generated by $\{\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\}$ is isomorphic to the graph group on the graph

$$\sigma_3^2 \text{—} \sigma_1^2 \text{—} \sigma_4^2 \text{—} \sigma_2^2$$

and the conjecture is established for the 5-string braid group.

References

[1] K. Appel and P. Schupp, *Artin groups and infinite Coxeter groups*, Invent. Math. 72 (1983), 201-220.

[2] J. Birman, *Braids, Links and Mapping Class Groups*, Annals of Mathematics Studies, No. 82, Princeton Univ. Press, Princeton, NJ, 1975.

[3] C. Droms, J. Lewin, and H. Servatius, *Tree groups and the 4 string pure braid group*, submitted.

[4] S. Pride, *On Tits' conjecture and other questions concerning Artin and generalized Artin groups*, Invent. Math. 86, no. 2 (1986), 347-356.