Critical groups of strongly regular graphs

Josh Ducey
James Madison University

Discrete Mathematics Seminar
University of Delaware

April 4, 2016

Outline

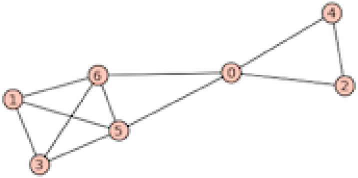
1. The critical group of a graph
2. Strongly regular graphs
3. Some examples

- Γ a simple graph

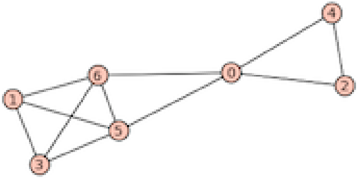
- Γ a simple graph
- A adjacency matrix

- Γ a simple graph
- A adjacency matrix
- $L = D - A$ Laplacian matrix

An example



An example



$$A = \begin{bmatrix} & & 1 & & 1 & 1 & 1 \\ & & & 1 & 1 & 1 & \\ 1 & & & & 1 & & \\ & 1 & & & & 1 & 1 \\ 1 & & 1 & & & & \\ 1 & 1 & & 1 & & & 1 \\ 1 & 1 & & 1 & & 1 & \end{bmatrix} \qquad L = \begin{bmatrix} 4 & & -1 & & -1 & -1 & -1 \\ & 3 & & -1 & -1 & -1 & \\ -1 & & 2 & & -1 & & \\ & -1 & & 3 & & -1 & -1 \\ -1 & & -1 & & 2 & & \\ -1 & -1 & & -1 & & 4 & -1 \\ -1 & -1 & & -1 & & -1 & 4 \end{bmatrix}$$

- Both A and L define a map $\mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$

- Both A and L define a map $\mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$
- $\mathrm{Coker}(A) = \mathbb{Z}^{V(\Gamma)} / \mathrm{Im}(A)$ is the *Smith group* of Γ

- Both A and L define a map $\mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$
- $\mathrm{Coker}(A) = \mathbb{Z}^{V(\Gamma)} / \mathrm{Im}(A)$ is the *Smith group* of Γ
- $\mathrm{Coker}(L) = \mathbb{Z}^{k} \oplus \mathcal{K}(\Gamma)$

- Both A and L define a map $\mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$
- $\mathrm{Coker}(A) = \mathbb{Z}^{V(\Gamma)} / \mathrm{Im}(A)$ is the *Smith group* of Γ
- $\mathrm{Coker}(L) = \mathbb{Z}^k \oplus \mathcal{K}(\Gamma)$
- $\mathcal{K}(\Gamma)$ is the *critical group* (or *sandpile group*)

Known critical groups

- trees, $\{0\}$
- n-cycle, Z_n
- complete graph K_n, $(Z_n)^{n-2}$
- wheel graph W_n (n odd), $(Z_{\ell_n})^2$
- line graphs (partial information)
- abelian Cayley graphs (partial information)
- Hypercube graph Q_n (2-part unknown)
- Payley, Peisert graphs
- many others

Smith normal form

- Start with a homomorphism of free abelian groups $M\colon\ \mathbb{Z}^n \to \mathbb{Z}^m$

Smith normal form

- Start with a homomorphism of free abelian groups $M\colon \mathbb{Z}^n \to \mathbb{Z}^m$
- There exist bases of domain and codomain so that M has matrix

$$\begin{bmatrix} s_1 & & & & & & \\ & s_2 & & & & & \\ & & s_3 & & & & \\ & & & \ddots & & & \\ & & & & 0 & & \\ & & & & & 0 & \\ & & & & & & \end{bmatrix},$$

where the s_i are integers with $s_i | s_{i+1}$ for all i.

Smith normal form

- Start with a homomorphism of free abelian groups $M\colon \mathbb{Z}^n \to \mathbb{Z}^m$
- There exist bases of domain and codomain so that M has matrix

$$\begin{bmatrix} s_1 & & & & & & \\ & s_2 & & & & & \\ & & s_3 & & & & \\ & & & \ddots & & & \\ & & & & 0 & & \\ & & & & & 0 & \end{bmatrix},$$

  where the s_i are integers with $s_i | s_{i+1}$ for all i.
- The s_i are called the invariant factors of M, and

$$\mathrm{Coker}(M) \cong \mathbb{Z}/s_1\mathbb{Z} \oplus \mathbb{Z}/s_2\mathbb{Z} \oplus \cdots$$

Outline

1. The critical group of a graph
2. Strongly regular graphs
3. Some examples

- Let Γ be an $srg(v, k, \lambda, \mu)$

- Let Γ be an $srg(v, k, \lambda, \mu)$
- $A^2 = kI + \lambda A + \mu(J - I - A)$

- Let Γ be an $srg(v, k, \lambda, \mu)$
- $A^2 = kI + \lambda A + \mu(J - I - A)$
- $L = kI - A$

- Let Γ be an $srg(v, k, \lambda, \mu)$
- $A^2 = kI + \lambda A + \mu(J - I - A)$
- $L = kI - A$

- Let Γ be an $srg(v, k, \lambda, \mu)$
- $A^2 = kI + \lambda A + \mu(J - I - A)$
- $L = kI - A$

$$L^2 + (\lambda - \mu - 2k)L = (k - k^2 + \lambda k - \mu - \mu k)I + \mu J$$

- Let Γ be an $srg(v, k, \lambda, \mu)$
- $A^2 = kI + \lambda A + \mu(J - I - A)$
- $L = kI - A$

$$L^2 + (\lambda - \mu - 2k)L = (k - k^2 + \lambda k - \mu - \mu k)I + \mu J$$

Lemma

If p is a prime and p^a exactly divides $k - k^2 + \lambda k - \mu - \mu k$, then p^a is an upper bound for the exponent of the p-primary component of $\mathcal{K}(\Gamma)$.

Proof

- $L\colon \mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$

Proof

- $L\colon \mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$
- Restrict L to the subgroup

$$Y = \left\{ \sum_{v \in V(\Gamma)} a_v v \;\middle|\; \sum_{v \in V(\Gamma)} a_v = 0 \right\}.$$

Proof

- $L\colon \mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$
- Restrict L to the subgroup

$$Y = \left\{ \sum_{v \in V(\Gamma)} a_v v \;\middle|\; \sum_{v \in V(\Gamma)} a_v = 0 \right\}.$$

- The previous equation becomes

$$(L + (\lambda - \mu - 2k)I)\, L = (k - k^2 + \lambda k - \mu - \mu k)I.$$

Proof

- $L\colon \mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$
- Restrict L to the subgroup

$$Y = \left\{ \sum_{v \in V(\Gamma)} a_v v \;\middle|\; \sum_{v \in V(\Gamma)} a_v = 0 \right\}.$$

- The previous equation becomes

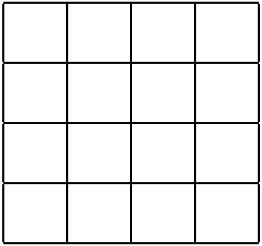
$$(L + (\lambda - \mu - 2k)I)\, L = (k - k^2 + \lambda k - \mu - \mu k)I.$$

- Consider SNF bases.

Outline

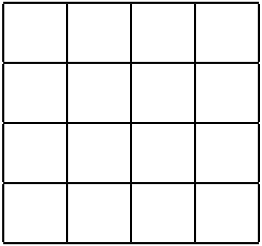
The rook's graph R_n

- Let R_n be the graph having vertex set the squares of an $n \times n$ grid.



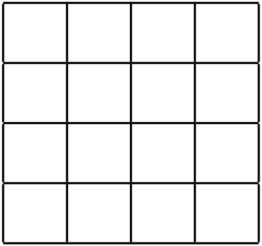
The rook's graph R_n

- Let R_n be the graph having vertex set the squares of an $n \times n$ grid.



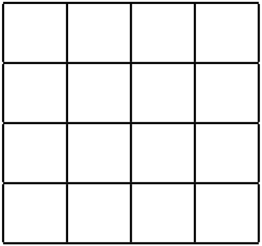
- Two squares are adjacent when they lie in the same row or column.

The rook's graph R_n



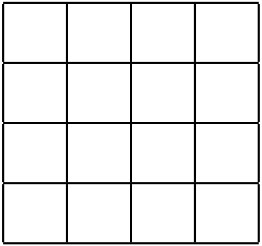
- $v = n^2$

The rook's graph R_n



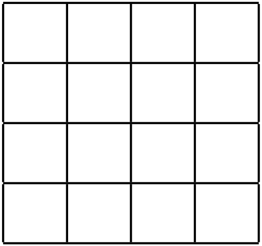
- $v = n^2$
- $k = 2(n-1)$

The rook's graph R_n



- $v = n^2$
- $k = 2(n-1)$
- $\lambda = n-2$

The rook's graph R_n



- $v = n^2$
- $k = 2(n-1)$
- $\lambda = n-2$
- $\mu = 2$

The rook's graph R_n

$$L^2 + (\lambda - \mu - 2k)L = (k - k^2 + \lambda k - \mu - \mu k)I + \mu J$$
$$L^2 + (-3n)L = (-2n^2)I + 2J$$

The rook's graph R_n

$$L^2 + (\lambda - \mu - 2k)L = (k - k^2 + \lambda k - \mu - \mu k)I + \mu J$$
$$L^2 + (-3n)L = (-2n^2)I + 2J$$

When n is odd, the 2-part of $\mathcal{K}(R_n)$ is elementary abelian.

The rook's graph R_n

$$L^2 + (\lambda - \mu - 2k)L = (k - k^2 + \lambda k - \mu - \mu k)I + \mu J$$
$$L^2 + (-3n)L = (-2n^2)I + 2J$$

When n is odd, the 2-part of $\mathcal{K}(R_n)$ is elementary abelian. In general,

$$\mathcal{K}(R_n) \cong (\mathbf{Z}_{2n})^{(n-2)^2+1} \oplus (\mathbf{Z}_{2n^2})^{2(n-2)}$$

Order of the $\mathcal{K}(\Gamma)$

- R_n is an SRG$(n^2, 2(n-1), n-2, 2)$

Order of the $\mathcal{K}(\Gamma)$

- R_n is an SRG$(n^2, 2(n-1), n-2, 2)$
- Its adjacency spectrum is

$$[-2]^{(n-1)^2}, [n-2]^{2n-2}, [2(n-1)]^1.$$

Order of the $\mathcal{K}(\Gamma)$

- R_n is an SRG$(n^2, 2(n-1), n-2, 2)$
- Its adjacency spectrum is

$$[-2]^{(n-1)^2}, [n-2]^{2n-2}, [2(n-1)]^1.$$

- $$\begin{aligned} |S(R_n)| &= 2^{(n-1)^2} \cdot (n-2)^{2n-2} \cdot 2(n-1) \\ &= 2^{(n-2)^2} \cdot (2(n-2))^{2n-3} \cdot 2(n-1)(n-2). \end{aligned}$$

Order of the $\mathcal{K}(\Gamma)$

- R_n is an $\mathrm{SRG}(n^2, 2(n-1), n-2, 2)$
- Its adjacency spectrum is
$$[-2]^{(n-1)^2}, [n-2]^{2n-2}, [2(n-1)]^1.$$
- $$\begin{aligned}|S(R_n)| &= 2^{(n-1)^2} \cdot (n-2)^{2n-2} \cdot 2(n-1) \\ &= 2^{(n-2)^2} \cdot (2(n-2))^{2n-3} \cdot 2(n-1)(n-2).\end{aligned}$$
- L has spectrum
$$[2n]^{(n-1)^2}, [n]^{2n-2}, [0]^1.$$

Order of the $\mathcal{K}(\Gamma)$

- R_n is an SRG$(n^2, 2(n-1), n-2, 2)$
- Its adjacency spectrum is

$$[-2]^{(n-1)^2}, [n-2]^{2n-2}, [2(n-1)]^1.$$

- $$\begin{aligned} |S(R_n)| &= 2^{(n-1)^2} \cdot (n-2)^{2n-2} \cdot 2(n-1) \\ &= 2^{(n-2)^2} \cdot (2(n-2))^{2n-3} \cdot 2(n-1)(n-2). \end{aligned}$$

- L has spectrum

$$[2n]^{(n-1)^2}, [n]^{2n-2}, [0]^1.$$

- Matrix tree theorem implies

$$\begin{aligned} |\mathcal{K}(R_n)| &= \frac{1}{n^2} \cdot (2n)^{(n-1)^2} \cdot n^{2n-2} \\ &= (2n)^{(n-2)^2+1} \cdot (2n^2)^{2(n-2)}. \end{aligned}$$

Lemma

Let G be a finite abelian group, generated by the elements $x_1, x_2, \ldots, x_k$. Suppose that there exist integers $r_1, r_2, \ldots, r_k$ so that $|G| = r_1 \cdot r_2 \cdots r_k$ and $|x_i|$ divides r_i, for $1 \leq i \leq k$. Then

$$G \cong \mathbf{Z}_{r_1} \oplus \mathbf{Z}_{r_2} \oplus \cdots \oplus \mathbf{Z}_{r_k}.$$

Chip-firing

- We label each vertex of the graph with an integer–we call the result a *configuration*

Chip-firing

- We label each vertex of the graph with an integer–we call the result a *configuration*
- Two configurations v_1 and v_2 represent the same element of the critical group if $v_1 - v_2 = Lu$, or

$$v_1 - Lu = v_2$$

We can get from v_1 to v_2 by *chip-firing*.

Chip-firing

- We label each vertex of the graph with an integer–we call the result a *configuration*
- Two configurations v_1 and v_2 represent the same element of the critical group if $v_1 - v_2 = Lu$, or

$$v_1 - Lu = v_2$$

  We can get from v_1 to v_2 by *chip-firing*.
- We may also restrict to configurations with vertices summing to zero.

Example: $\mathcal{K}(R_4)$

-8	8		
8	-8		

Example: $\mathcal{K}(R_4)$

-8	8		
8	-8		

Example: $\mathcal{K}(R_4)$

-8	8		
8	-8		

$\Rightarrow$

-2	7	-1	-1
7	-8		
-1			
-1			

Example: $\mathcal{K}(R_4)$

-8	8		
8	-8		

$\Rightarrow$

-2	7	-1	-1
7	-8		
-1			
-1			

$\Rightarrow$

-1	1		
7	-7		
-1	1		
-1	1		

Example: $\mathcal{K}(R_4)$

-8	8		
8	-8		

$\Rightarrow$

-2	7	-1	-1
7	-8		
-1			
-1			

$\Rightarrow$

-1	1		
7	-7		
-1	1		
-1	1		

$\Rightarrow$

-1			
6	-1	-1	-1
-1			
-1			

Example: $\mathcal{K}(R_4)$

-8	8		
8	-8		

⇒

-2	7	-1	-1
7	-8		
-1			
-1			

⇒

-1	1		
7	-7		
-1	1		
-1	1		

⇒

-1			
6	-1	-1	-1
-1			
-1			

⇒

Example: $\mathcal{K}(R_4)$

The order of the critical group, $\mathcal{K}(R_4)$, is $2^{35} = 34359738368$.

Example: $\mathcal{K}(R_4)$

The order of the critical group, $\mathcal{K}(R_4)$, is $2^{35} = 34359738368$.

$$\mathcal{K}(R_4) \cong (\mathbf{Z}_8)^5 \oplus (\mathbf{Z}_{32})^4 \,.$$

Example: $\mathcal{K}(R_4)$

-1	1		
1	-1		

-1	1		
1	-1		

-1		1	
1		-1	

-1		1	
1		-1	

-1	1		

-1		1	

-1			
1			

-1			
1			

-3	1	1	1

Example: $\mathcal{K}(R_4)$

-32	32		

Example: $\mathcal{K}(R_4)$

-32	32		

Example: $\mathcal{K}(R_4)$

-32	32		

$\Rightarrow$

-27	2	5	5
	5		
	5		
	5		

Example: $\mathcal{K}(R_4)$

-32	32		

⇒

-27	2	5	5
	5		
	5		
	5		

⇒

3	-3		
-5	5		
-5	5		
-5	5		

Example: $\mathcal{K}(R_4)$

-32	32		

$\Rightarrow$

-27	2	5	5
	5		
	5		
	5		

$\Rightarrow$

3	-3		
-5	5		
-5	5		
-5	5		

$\Rightarrow$

3			
-4	1	1	1
-4	1	1	1
-4	1	1	1

Example: $\mathcal{K}(R_4)$

-32	32		

$\Rightarrow$

-27	2	5	5
	5		
	5		
	5		

$\Rightarrow$

3	-3		
-5	5		
-5	5		
-5	5		

$\Rightarrow$

3			
-4	1	1	1
-4	1	1	1
-4	1	1	1

$\Rightarrow$

Example: $\mathcal{K}(R_4)$

These elements generate the group.

-1	1		
1	-1		

-1	1		
1	-1		

-1		1	
1		-1	

-1		1	
1		-1	

-1	1		

-1		1	

-1			
1			

-1			
1			

-3	1	1	1

Example: $\mathcal{K}(R_4)$

-3	1	1	1

+

1	-1		

+

1		-1	

=

-1			1

Example: $\mathcal{K}(R_4)$

-3	1	1	1

$+$

1	-1		

$+$

1		-1	

$=$

-1			1

-6	1	1	1
1			
1			
1			

$+$

3	-1	-1	-1

$+$

1			
-1			

$+$

1			
-1			

$=$

-1			
1			

A similar game can be played with the adjacency matrix, and with the graph ${R_n}^c$.

A similar game can be played with the adjacency matrix, and with the graph ${R_n}^c$.

Theorem (D, Gerhard, Watson)

The critical group and Smith group of R_n and its complement R_n^c are given by the following isomorphisms:

$$\mathcal{K}(R_n) \cong (\mathbf{Z}_{2n})^{(n-2)^2+1} \oplus (\mathbf{Z}_{2n^2})^{2(n-2)}$$

$$S(R_n) \cong (\mathbf{Z}_2)^{(n-2)^2} \oplus \left(\mathbf{Z}_{2(n-2)}\right)^{2n-3} \oplus \mathbf{Z}_{2(n-1)(n-2)}$$

$$\mathcal{K}(R_n^c) \cong \left(\mathbf{Z}_{n(n-2)}\right)^{(n-2)^2-1} \oplus \left(\mathbf{Z}_{n(n-1)(n-2)}\right)^2 \oplus \left(\mathbf{Z}_{n^2(n-1)(n-2)}\right)^{2(n-2)}$$

$$S(R_n^c) \cong \left(\mathbf{Z}_{(n-1)}\right)^{2(n-1)} \oplus \mathbf{Z}_{(n-1)^2}.$$

Another Example.

Another Example.

Suppose that Γ is an $srg(3250, 57, 0, 1)$.

Another Example.

Suppose that Γ is an $srg(3250, 57, 0, 1)$.
L has spectrum $[50]^{1729}, [65]^{1520}, [0]^{1}$, so

$$\begin{aligned} |\mathcal{K}(\Gamma)| &= \frac{1}{3250} \cdot 50^{1729} \cdot 65^{1520} \\ &= 2^{1728} \cdot 5^{4975} \cdot 13^{1519}. \end{aligned}$$

Another Example.

Suppose that Γ is an *srg*(3250, 57, 0, 1).
L has spectrum $[50]^{1729}, [65]^{1520}, [0]^{1}$, so

$$|\mathcal{K}(\Gamma)| = \frac{1}{3250} \cdot 50^{1729} \cdot 65^{1520}$$
$$= 2^{1728} \cdot 5^{4975} \cdot 13^{1519}.$$

$$L^2 + (\lambda - \mu - 2k)L = (k - k^2 + \lambda k - \mu - \mu k)I + \mu J$$
$$L^2 + (-115)L = -(2 \cdot 5^3 \cdot 13)I + J$$

Another Example.

Suppose that Γ is an $srg(3250, 57, 0, 1)$.
L has spectrum $[50]^{1729}, [65]^{1520}, [0]^1$, so

$$|\mathcal{K}(\Gamma)| = \frac{1}{3250} \cdot 50^{1729} \cdot 65^{1520} = 2^{1728} \cdot 5^{4975} \cdot 13^{1519}.$$

$$L^2 + (\lambda - \mu - 2k)L = (k - k^2 + \lambda k - \mu - \mu k)I + \mu J$$
$$L^2 + (-115)L = -(2 \cdot 5^3 \cdot 13)I + J$$

$$K(\Gamma) \cong (\mathbf{Z}/2\mathbf{Z})^{1728} \oplus (\mathbf{Z}/13\mathbf{Z})^{1519} \oplus (\mathbf{Z}/5\mathbf{Z})^{e_1} \oplus \left(\mathbf{Z}/5^2\mathbf{Z}\right)^{e_2} \oplus \left(\mathbf{Z}/5^3\mathbf{Z}\right)^{e_3}$$

Localization

$$L\colon\ \mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$$

Localization

$$L\colon\ \mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$$

$$L\colon\ {\mathbb{Z}_p}^{V(\Gamma)} \to {\mathbb{Z}_p}^{V(\Gamma)}$$

Localization

$$L\colon\ \mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$$

$$L\colon\ \mathbb{Z}_p{}^{V(\Gamma)} \to \mathbb{Z}_p{}^{V(\Gamma)}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Localization

- $L\colon {\mathbb{Z}_p}^{V(\Gamma)} \to {\mathbb{Z}_p}^{V(\Gamma)}$
- $M_i = \left\{x \in {\mathbb{Z}_p}^n \mid Lx \in p^i\, {\mathbb{Z}_p}^m\right\}$
- $N_i = \left\{p^{-i} Lx \,|\, x \in M_i\right\}$
- Let e_i denote multiplicity of p^i in SNF
- $$\dim_{\mathbb{F}_p} \overline{M_i} = \dim_{\mathbb{F}_p} \overline{\ker(L)} + e_i + e_{i+1} + \cdots$$
  and
  $$\dim_{\mathbb{F}_p} \overline{N_i} = e_0 + e_1 + \cdots + e_i.$$

Consider the inclusions of the eigenspaces of L in these modules:

- $V_{65} \cap \mathbf{Z}_5^{V(\Gamma)} \subseteq N_1$, and so $\overline{V_{65} \cap \mathbf{Z}_5^{V(\Gamma)}} \subseteq \overline{N_1}$

Consider the inclusions of the eigenspaces of L in these modules:

- $V_{65} \cap \mathbf{Z}_5^{V(\Gamma)} \subseteq N_1$, and so $\overline{V_{65} \cap \mathbf{Z}_5^{V(\Gamma)}} \subseteq \overline{N_1}$
- $V_{50} \cap \mathbf{Z}_5^{V(\Gamma)} \subseteq M_2$

We get the inequalities:

$$\begin{aligned} 1520 &\leq e_0 + e_1 \\ 1729 &\leq 1 + e_2 + e_3. \end{aligned}$$

Case 1: $1520 = e_0 + e_1$ and $1729 = e_2 + e_3$.

Case 2: $1521 = e_0 + e_1$ and $1728 = e_2 + e_3$.

We get the inequalities:

$$1520 \leq e_0 + e_1$$
$$1729 \leq 1 + e_2 + e_3.$$

Case 1: $1520 = e_0 + e_1$ and $1729 = e_2 + e_3$.

Case 2: $1521 = e_0 + e_1$ and $1728 = e_2 + e_3$.

We also know $|Syl_5(K(\Gamma))| = 5^{4975}$, so

$$4975 = e_1 + 2e_2 + 3e_3.$$

Theorem

Let Γ *be an* $srg(3250, 57, 0, 1)$. *Let* e_0 *denote the rank of the Laplacian matrix of* Γ *over a field of characteristic* 5. *Then either*

$$Syl_5(K(\Gamma)) \cong (\mathbf{Z}/5\mathbf{Z})^{1520-e_0} \oplus \left(\mathbf{Z}/5^2\mathbf{Z}\right)^{1732-e_0} \oplus \left(\mathbf{Z}/5^3\mathbf{Z}\right)^{e_0-3}$$

or

$$Syl_5(K(\Gamma)) \cong (\mathbf{Z}/5\mathbf{Z})^{1521-e_0} \oplus \left(\mathbf{Z}/5^2\mathbf{Z}\right)^{1730-e_0} \oplus \left(\mathbf{Z}/5^3\mathbf{Z}\right)^{e_0-2}.$$

What is the 5-rank of L?

Thank you for your attention!