The Smith group and the critical group of the Grassmann graph of lines in a finite projective space.

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This is joint work with Peter Sin.

Outline

- Γ, graph with vertices the 2-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$ ($q = p^t$), adjacent when far apart
- A adjacency matrix, $L = D - A$ Laplacian matrix
- Complement graph denoted Γ'
- A', L'
- Strongly regular with parameters

$$v' = \begin{bmatrix} n \\ 2 \end{bmatrix}_q$$
$$k' = q(q+1)\begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q$$
$$\lambda' = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + q^2 - 2$$
$$\mu' = (q+1)^2.$$

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- $|K(\Gamma)|$ counts number of spanning trees

Elementary divisors

- $A\colon \mathbb{Z}_p^{V(\Gamma)} \to \mathbb{Z}_p^{V(\Gamma)}$
- $M_i = \left\{x \in \mathbb{Z}_p^{V(\Gamma)} \mid Ax \in p^i\,\mathbb{Z}_p^{V(\Gamma)}\right\}$
- $N_i = \left\{p^{-i}Ax \,|\, x \in M_i\right\}$
- Let e_i denote multiplicity of p^i as elementary divisor of A
- $$\dim_{\mathbb{F}_p} \overline{M_i} = \dim_{\mathbb{F}_p} \overline{\ker(A)} + e_i + e_{i+1} + \cdots$$

  and

  $$\dim_{\mathbb{F}_p} \overline{N_i} = e_0 + e_1 + \cdots + e_i.$$

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- $A|_Y[A|_Y - qz_1 I] = q^3 z_2 I$
- A has spectrum

$$q^4 \begin{bmatrix} n-2 \\ 2 \end{bmatrix}_q,\ -q^2 \begin{bmatrix} n-3 \\ 1 \end{bmatrix}_q,\ q$$

with respective multiplicities $1, \begin{bmatrix} n \\ 1 \end{bmatrix}_q - 1, \begin{bmatrix} n \\ 2 \end{bmatrix}_q - \begin{bmatrix} n \\ 1 \end{bmatrix}_q$.

Skew lines, A

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- $$\left[{n \atop 1}\right]_q - 1 \leq e_{2t} + \cdots + e_{3t} \quad (1)$$
  $$\left[{n \atop 2}\right]_q - \left[{n \atop 1}\right]_q \leq e_0 + \cdots + e_t. \quad (2)$$

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- In general, if $A_{r,s}$ denotes the zero-intersection incidence matrix between r-spaces and s-spaces, then we have that
  $$-A_{r,s} \equiv A_{r,1}A_{1,s} \pmod{p^t}$$

Skew lines, A

Theorem (Brouwer-D-Sin 2011)

Let e_i denote the multiplicity of p^i as a p-adic elementary divisor of $A_{2,1}A_{1,2}$.

1. $e_{4t} = 1$.
2. *For $i \neq 4t$,*

$$e_i = \sum_{\vec{s} \in \Gamma(i)} d(\vec{s}),$$

where

$$\Gamma(i) = \bigcup_{\substack{\alpha+\beta=i \\ 0 \leq \alpha \leq t \\ 0 \leq \beta \leq t}} {}_{\beta}\mathcal{H} \cap \mathcal{H}_{\alpha}.$$

Summation over an empty set is interpreted to result in 0.

- L: same result as for A, but no e_{4t}.
- A': p-part is cyclic of order p^t
- L': no p-part

Notation: The vector space has dimension n over a field of $q = p^t$ elements.

(n, p, t)	matrix	(elem. div. : multiplicity)
(4, 2, 1)	A	$(2 : 14)$, $(2^2 : 8)$, $(2^3 : 6)$, $(2^4 : 1)$
	L	$(2 : 14)$, $(2^2 : 8)$, $(2^3 : 6)$ $(5 : 13)$ $(7 : 19)$
	A'	$(2 : 1)$ $(3 : 8)$, $(3^2 : 14)$
	L'	$(3 : 8)$, $(3^2 : 13)$ $(5 : 13)$ $(7 : 19)$

Notation: The vector space has dimension n over a field of $q = p^t$ elements.

(n, p, t)	matrix	(elem. div. : multiplicity)
(4, 2, 2)	A	$(2:16), (2^2:220), (2^4:32), (2^5:16), (2^6:36), (2^8:1)$
	L	$(2:16), (2^2:220), (2^4:32), (2^5:16), (2^6:36)$ $(3:1), (3^2:271)$ $(7:271)$ $(17:83)$
	A'	$(2^2:1)$ $(3:84)$ $(5:190), (5^2:84)$
	L'	$(3:271)$ $(5:190), (5^2:83)$ $(7:271)$; $(17:83)$

p' part

- ℓ, a prime different than p
- Structure of $\mathbb{F}_\ell \, \mathrm{GL}(n, q)$-permutation module on 2-spaces is able to be understood. (few composition factors)
- We rely heavily on work of G. James
- $\mathbb{F}_\ell^{V(\Gamma)}$ has descending filtration with subquotients as Specht modules
- Straightforward arithmetic conditions determine the composition factors and multiplicities of the Specht modules

An example: Grassmann, L'

Parameters:

$$k' = q(q+1)\begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q, \quad \mu' = (q+1)^2, \quad \lambda' = \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + q^2 - 2,$$

$$r = q(q+1)\begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q - q^2\begin{bmatrix} n-3 \\ 1 \end{bmatrix}_q + 1 = \begin{bmatrix} n \\ 1 \end{bmatrix}_q,$$

$$s = q(q+1)\begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q + (q+1) = (q+1)\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q.$$

$$|K(\Gamma')| = \frac{\begin{bmatrix} n \\ 1 \end{bmatrix}_q^{f}\left((q+1)\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q\right)^g}{\begin{bmatrix} n \\ 2 \end{bmatrix}_q}$$

$$= \begin{bmatrix} n \\ 1 \end{bmatrix}_q^{f-1}(q+1)^{g+1}\begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q^{g-1}.$$

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- $|K(\Gamma')| = \left[{n \atop 1}\right]_q^{f-1} (q+1)^{g+1} \left[{n-1 \atop 1}\right]_q^{g-1}.$

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- $|K(\Gamma')| = {\genfrac{[}{]}{0pt}{}{n}{1}_q}^{f-1}(q+1)^{g+1}{\genfrac{[}{]}{0pt}{}{n-1}{1}_q}^{g-1}$.
- Case: $\ell \nmid q+1$, $\ell \mid \genfrac{[}{]}{0pt}{}{n}{1}_q$, and $\ell \nmid \genfrac{[}{]}{0pt}{}{n-1}{1}_q$

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- $|K(\Gamma')| = \left[{n \atop 1}\right]_q^{f-1}(q+1)^{g+1}\left[{n-1 \atop 1}\right]_q^{g-1}$.
- Case: $\ell \nmid q+1$, $\ell \mid \left[{n \atop 1}\right]_q$, and $\ell \nmid \left[{n-1 \atop 1}\right]_q$
- Let $a = v_\ell(\left[{n \atop 1}\right]_q)$. Then $v_\ell(|K(\Gamma')|) = a(f-1)$.

An example: Grassmann, L'

We have

$$f \leq \dim_F \overline{M}_a = 1 + \sum_{i \geq a} e_i,$$

Therefore,

$$a(f-1) = v_\ell(|K(\Gamma')|) = \sum_{i \geq 0} ie_i \geq \sum_{i \geq a} ie_i \geq a \sum_{i \geq a} e_i \geq a(f-1)$$

Since we must have equality throughout, it follows that $e_i = 0$ unless $i = 0$ or a, and that $e_a = f - 1$. Then as $\sum_{i \geq 0} e_i = g + f$, we have $e_0 = g + 1$.

Thank you for your attention!