The Critical Group of the Kneser graph KG(n, 2)

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The Kneser graphs Critical group of a graph Some Examples

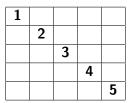
This is joint work with Ian Hill.

Outline

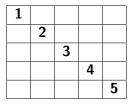
- 1 The Kneser graphs
- 2 Critical group of a graph
- Some Examples

- We let KG(n, k) denote the graph with vertices the size k subsets of an n element set.
- A pair of subsets are adjacent if and only if they are disjoint.

$K\overline{G(5,2)}$



KG(5,2)



We will compute the *critical group* of KG(n, 2).

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Critical group

• To any finite graph Γ we can attach an abelian group.

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- Chip-firing: attractive, elementary.

1						
	2					
		3				
			4			
				5		
					6	
						7

1	9					-9
	2				-9	
		3				
			4			
				5		
					6	9
						7

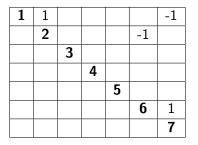
1	-1					-9
	2				-9	
		3	1	1	1	1
			4	1	1	1
				5	1	1
					6	10
						7

1		1	1	1		-9
	2	1	1	1	-9	
		3	2	2	1	1
			4	2	1	1
				5	1	1
					6	
						7

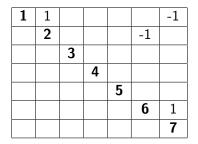
1						-10
	2	1	1	1	1	
		3	1	1	1	
			4	1	1	
				5	1	
					6	
						7

Example: $\overline{KG}(7, 2)$

1						
	2					
		3				
			4			
				5		
					6	
						7



This shows that the configuration above represents an element of the critical group with order dividing 9.



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$$\mathbf{Z}_3 \oplus (\mathbf{Z}_9)^7 \oplus \mathbf{Z}_{18} \oplus (\mathbf{Z}_{126})^5$$



Critical group

- ullet To any finite graph Γ we can attach an abelian group.
- Chip-firing: attractive, elementary.
- $L: \mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$

$$\mathbb{Z}^{V(\Gamma)}/\operatorname{Im}(L) = \mathcal{K}(\Gamma) \oplus \mathbb{Z}$$

(SNF)



Previous work on KG(n, 2)

- A_{comp} Brouwer and van Eijl (elementary row/col ops, 1993)
- L_{comp} Berget, et al. (critical groups of line graphs, 2012)
- A Wilson (SNF bases, 1990)

$$L\colon\thinspace \mathbb{Z}^{V(\Gamma)}\to \mathbb{Z}^{V(\Gamma)}$$

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$$\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 6 & \\ & & & 12 \\ & & & 0 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 4 & \\ & & & 0 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 3 & \\ & & & 3 & \\ & & & & 0 \end{pmatrix}$$

$$L \colon \mathbb{Z}_{p}^{V(\Gamma)} \to \mathbb{Z}_{p}^{V(\Gamma)}$$

- $L: \mathbb{Z}_p^{V(\Gamma)} \to \mathbb{Z}_p^{V(\Gamma)}$
- $M_i = \left\{ x \in \mathbb{Z}_p^{V(\Gamma)} \mid Lx \in p^i \, \mathbb{Z}_p^{V(\Gamma)} \right\}$
- $\mathbb{Z}_p^{V(\Gamma)} = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$
- $\mathbb{F}_{\mathsf{p}}^{V(\Gamma)} = \overline{M_0} \supseteq \overline{M_1} \supseteq \overline{M_2} \supseteq \cdots$
- Let e_i denote multiplicity of p^i in SNF
- $\dim_{\mathbb{F}_p} \overline{M_i} = \dim_{\mathbb{F}_p} \overline{\ker(L)} + e_i + e_{i+1} + \cdots$

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Let $\Gamma = KG(n, 2)$. Matrix-tree theorem gives us:

$$\begin{split} |\,\mathcal{K}(\Gamma)| &= \frac{\left[\frac{n(n-3)}{2}\right]^f \left[\frac{(n-4)(n-1)}{2}\right]^g}{\frac{n(n-1)}{2}} \\ &= \frac{n^{f-1}(n-1)^{g-1}(n-3)^f(n-4)^g}{2^{f+g-1}}, \end{split}$$

where f = n - 1 and g = n(n - 3)/2.

Case: $p \neq 2, p \nmid n, p \mid n - 3$

$$|\mathcal{K}(\Gamma)| = \frac{n^{f-1}(n-1)^{g-1}(n-3)^f(n-4)^g}{2^{f+g-1}}$$

Say $p^a \parallel n-3$. We have

$$af = v_p |\mathcal{K}(\Gamma)| = \sum_{i \geq 0} ie_i \geq \sum_{i \geq a} ie_i \geq a \sum_{i \geq a} e_i$$

$$= a \left(\dim \overline{M_a} - 1 \right)$$

$$\geq a \left((f+1) - 1 \right)$$

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Equality throughout. Follows that $e_a = f$, $e_0 = g$, $e_i = 0$ otherwise.



Case: $p \neq 2, p \mid n-1, p \mid n-4$ (So p = 3.)

$$|\mathcal{K}(\Gamma)| = \frac{n^{f-1}(n-1)^{g-1}(n-3)^f(n-4)^g}{2^{f+g-1}}$$

Say $p^a \parallel n - 1(a > 1)$. We have

$$\begin{split} a(g-1) + g &= v_p |\, \mathcal{K}(\Gamma)| = \sum_{i \geq 0} i e_i = \sum_{a+1 > i \geq 0} i e_i + \sum_{i \geq a+1} i e_i \\ &\geq \sum_{a+1 > i \geq 0} e_i + (a+1) \sum_{i \geq a+1} e_i \\ &= \left(\dim \overline{M_1} - \dim \overline{M_{a+1}} \right) + (a+1) \left(\dim \overline{M_{a+1}} - 1 \right) \\ &= \dim \overline{M_1} + a \dim \overline{M_{a+1}} - a - 1 \\ &\geq (g+1) + ag - a - 1 \end{split}$$

Equality throughout forces: $e_{a+1} = g - 1$, $e_1 = 1$, $e_0 = f$.

Case:
$$p \neq 2, p \mid n - 1, p \mid n - 4 \text{ (So } p = 3.) \text{ cont'd}$$

We needed: $\dim \overline{M_{a+1}} \ge g$ and $\dim \overline{M_1} \ge g+1$.

Case:
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$$F_{\mathsf{p}}^{V(\Gamma)} \cong D_2 \oplus D_1$$
 F

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$$F$$

$$\mathbb{F}_{\mathsf{p}}^{\ V(\Gamma)} \cong D_2 \oplus D_1$$
 F

Theory of these \mathbb{F}_p S_n -modules due to G. James (1980s). Generalizes to skew lines in PG(n, q).

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Thank you for your attention!