The Critical Group of the Kneser graph $KG(n, 2)$

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This is joint work with Ian Hill.

Outline

- We let $KG(n,k)$ denote the graph with vertices the size k subsets of an n element set.
- A pair of subsets are adjacent if and only if they are disjoint.

$KG(5, 2)$

1				
	2			
		3		
			4	
				5

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1				
	2			
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We will compute the *critical group* of $KG(n,2)$.

Outline

Critical group

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- Chip-firing: attractive, elementary.

Example: $KG(7,2)$

1						
	2					
		3				
			4			
				5		
					6	
						7

Example: $KG(7,2)$

1	9					-9
	2				-9	
		3				
			4			
				5		
					6	9
						7

Example: $KG(7,2)$

1	-1					-9
	2				-9	
		3	1	1	1	1
			4	1	1	1
				5	1	1
					6	10
						7

Example: $KG(7,2)$

1		1	1	1		-9
	2	1	1	1	-9	
		3	2	2	1	1
			4	2	1	1
				5	1	1
					6	
						7

Example: $KG(7,2)$

1						-10
	2	1	1	1	1	
		3	1	1	1	
			4	1	1	
				5	1	
					6	
						7

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1						
	2					
		3				
			4			
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		3				
			4			
				5		
					6	1
						7

This shows that the configuration above represents an element of the critical group with order dividing 9.

Example: $KG(7,2)$

1	1					-1
	2				-1	
		3				
			4			
				5		
					6	1
						7

This shows that the configuration above represents an element of the critical group with order dividing 9.

$\mathbf{Z}_3 \oplus (\mathbf{Z}_9)^7 \oplus \mathbf{Z}_{18} \oplus (\mathbf{Z}_{126})^5$

Critical group

- To any finite graph Γ we can attach an abelian group.
- Chip-firing: attractive, elementary.
- $L\colon \mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$

$$\mathbb{Z}^{V(\Gamma)}/\operatorname{Im}(L) = \mathcal{K}(\Gamma) \oplus \mathbb{Z}$$

(SNF)

Previous work on $KG(n,2)$

- A_{comp} - Brouwer and van Eijl (elementary row/col ops, 1993)
- L_{comp} - Berget, et al. (critical groups of line graphs, 2012)
- A - Wilson (SNF bases, 1990)

Localization

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$$L\colon {\mathbb{Z}_p}^{V(\Gamma)} \to {\mathbb{Z}_p}^{V(\Gamma)}$$

Localization

- $L\colon \mathbb{Z}_p{}^{V(\Gamma)} \to \mathbb{Z}_p{}^{V(\Gamma)}$
- $M_i = \left\{ x \in \mathbb{Z}_p{}^{V(\Gamma)} \mid Lx \in p^i\, \mathbb{Z}_p{}^{V(\Gamma)} \right\}$
- $\mathbb{Z}_p{}^{V(\Gamma)} = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$
- $\mathbb{F}_p{}^{V(\Gamma)} = \overline{M_0} \supseteq \overline{M_1} \supseteq \overline{M_2} \supseteq \cdots$
- Let e_i denote multiplicity of p^i in SNF
- $\dim_{\mathbb{F}_p} \overline{M_i} = \dim_{\mathbb{F}_p} \overline{\ker(L)} + e_i + e_{i+1} + \cdots$

Outline

Let $\Gamma = KG(n,2)$. Matrix-tree theorem gives us:

$$|\mathcal{K}(\Gamma)| = \frac{\left[\frac{n(n-3)}{2}\right]^f \left[\frac{(n-4)(n-1)}{2}\right]^g}{\frac{n(n-1)}{2}}$$
$$= \frac{n^{f-1}(n-1)^{g-1}(n-3)^f(n-4)^g}{2^{f+g-1}},$$

where $f = n-1$ and $g = n(n-3)/2$.

Case: $p \neq 2, p \nmid n, p \mid n-3$

$$|\mathcal{K}(\Gamma)| = \frac{n^{f-1}(n-1)^{g-1}(n-3)^f(n-4)^g}{2^{f+g-1}}$$

Say $p^a \parallel n-3$. We have

$$\begin{aligned} af = v_p|\mathcal{K}(\Gamma)| = \sum_{i\geq 0} ie_i \geq \sum_{i\geq a} ie_i \geq a\sum_{i\geq a} e_i \\ = a\left(\dim \overline{M_a} - 1\right) \\ \geq a\left((f+1)-1\right) \\ = af. \end{aligned}$$

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Equality throughout. Follows that $e_a = f$, $e_0 = g$, $e_i = 0$ otherwise.

Case: $p \neq 2, p \mid n-1, p \mid n-4$ (So $p = 3$.)

$$|\mathcal{K}(\Gamma)| = \frac{n^{f-1}(n-1)^{g-1}(n-3)^f(n-4)^g}{2^{f+g-1}}$$

Say $p^a \parallel n-1 (a > 1)$. We have

$$\begin{aligned}
a(g-1)+g = v_p|\mathcal{K}(\Gamma)| = \sum_{i\geq 0} ie_i &= \sum_{a+1>i\geq 0} ie_i + \sum_{i\geq a+1} ie_i \\
&\geq \sum_{a+1>i\geq 0} e_i + (a+1)\sum_{i\geq a+1} e_i \\
&= \left(\dim \overline{M_1} - \dim \overline{M_{a+1}}\right) + (a+1)\left(\dim \overline{M_{a+1}} - 1\right) \\
&= \dim \overline{M_1} + a \dim \overline{M_{a+1}} - a - 1 \\
&\geq (g+1) + ag - a - 1
\end{aligned}$$

Equality throughout forces: $e_{a+1} = g-1, e_1 = 1, e_0 = f$.

Case: $p \neq 2, p \mid n-1, p \mid n-4$ (So $p=3$.) cont'd

We needed: $\dim \overline{M_{a+1}} \geq g$ and $\dim \overline{M_1} \geq g+1$.

Case: $p \neq 2, p \mid n-1, p \mid n-4$ (So $p = 3$.) cont'd

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$$\begin{array}{ccccc} & & F & & \\ {\mathbb{F}_p}^{V(\Gamma)} & \cong & D_2 & \oplus & D_1 \\ & & F & & \end{array}$$

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Theory of these $\mathbb{F}_p S_n$-modules due to G. James (1980s).
Generalizes to skew lines in $PG(n, q)$.

Thank you for your attention!