Critical groups of strongly regular graphs

Josh Ducey
James Madison University, USA

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Kyoto University

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Outline

- Γ a simple graph

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- A adjacency matrix

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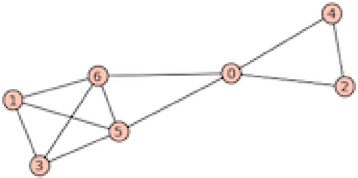
- $L = D - A$ Laplacian matrix

An example



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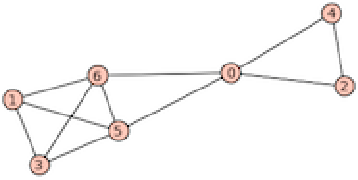

$$A = \begin{bmatrix} & & 1 & & 1 & 1 & 1 \\ & & & 1 & 1 & 1 & \\ 1 & & & & 1 & & \\ & 1 & & & & 1 & 1 \\ 1 & & 1 & & & & \\ 1 & 1 & & 1 & & & 1 \\ 1 & 1 & & 1 & & 1 & \end{bmatrix} \quad L = \begin{bmatrix} 4 & & -1 & & -1 & -1 & -1 \\ & 3 & & -1 & -1 & -1 & \\ -1 & & 2 & & -1 & & \\ & -1 & & 3 & & -1 & -1 \\ -1 & & -1 & & 2 & & \\ -1 & -1 & & -1 & & 4 & -1 \\ -1 & -1 & & -1 & & -1 & 4 \end{bmatrix}$$

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- $\mathrm{Coker}(L) = \mathbb{Z}^k \oplus \mathcal{K}(\Gamma)$
- $\mathcal{K}(\Gamma)$ is the *critical group* (or *sandpile group*)

Known critical groups

- trees, $\{0\}$
- n-cycle, Z_n
- complete graph K_n, $(Z_n)^{n-2}$
- wheel graph W_n (n odd), $(Z_{\ell_n})^2$
- line graphs (partial information)
- abelian Cayley graphs (partial information)
- Hypercube graph Q_n (2-part unknown)
- Payley, Peisert graphs
- many others

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where the s_i are integers with $s_i | s_{i+1}$ for all i.

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  where the s_i are integers with $s_i | s_{i+1}$ for all i.
- The s_i are called the invariant factors of M, and

$$\mathrm{Coker}(M) \cong \mathbb{Z}/s_1\mathbb{Z} \oplus \mathbb{Z}/s_2\mathbb{Z} \oplus \cdots$$

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Lemma

If p is a prime and p^a exactly divides $k - k^2 + \lambda k - \mu - \mu k$, then p^a is an upper bound for the exponent of the p-primary component of $\mathcal{K}(\Gamma)$.

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- Consider SNF bases.

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The rook's graph R_n

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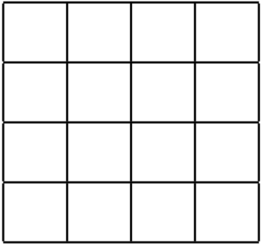

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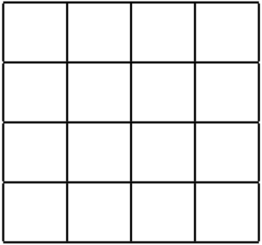

- Two squares are adjacent when they lie in the same row or column.

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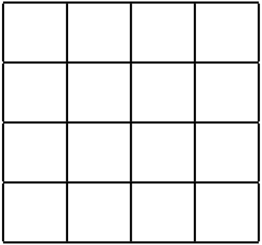

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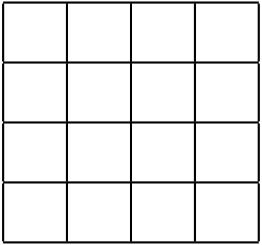

- $v = n^2$
- $k = 2(n-1)$

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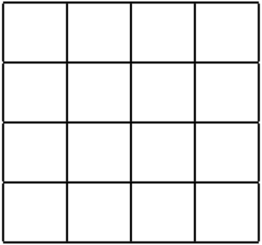

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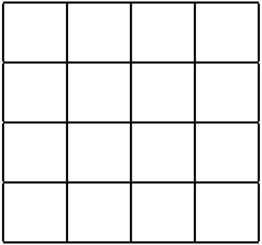

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$$\begin{aligned} L^2 + (\lambda - \mu - 2k)L &= (k - k^2 + \lambda k - \mu - \mu k)I + \mu J \\ L^2 + (-3n)L &= (-2n^2)I + 2J \end{aligned}$$

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When n is odd, the 2-part of $\mathcal{K}(R_n)$ is elementary abelian. In general,

$$\mathcal{K}(R_n) \cong (\mathbf{Z}_{2n})^{(n-2)^2+1} \oplus (\mathbf{Z}_{2n^2})^{2(n-2)}$$

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L has spectrum $[50]^{1729}, [65]^{1520}, [0]^{1}$, so

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$$= 2^{1728} \cdot 5^{4975} \cdot 13^{1519}.$$

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$$K(\Gamma) \cong (\mathbf{Z}/2\mathbf{Z})^{1728} \oplus (\mathbf{Z}/13\mathbf{Z})^{1519} \oplus (\mathbf{Z}/5\mathbf{Z})^{e_1} \oplus \left(\mathbf{Z}/5^2\mathbf{Z}\right)^{e_2} \oplus \left(\mathbf{Z}/5^3\mathbf{Z}\right)^{e_3}$$

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$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

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Localization

- $L\colon {\mathbb{Z}_p}^{V(\Gamma)} \to {\mathbb{Z}_p}^{V(\Gamma)}$
- $M_i = \left\{ x \in {\mathbb{Z}_p}^n \mid Lx \in p^i\, {\mathbb{Z}_p}^m \right\}$
- $N_i = \left\{ p^{-i} Lx \,|\, x \in M_i \right\}$
- Let e_i denote multiplicity of p^i in SNF
- $$\dim_{\mathbb{F}_p} \overline{M_i} = \dim_{\mathbb{F}_p} \overline{\ker(L)} + e_i + e_{i+1} + \cdots$$
  and
  $$\dim_{\mathbb{F}_p} \overline{N_i} = e_0 + e_1 + \cdots + e_i.$$

Consider the inclusions of the eigenspaces of L in these modules:

- $V_{65} \cap \mathbf{Z}_5^{V(\Gamma)} \subseteq N_1$, and so $\overline{V_{65} \cap \mathbf{Z}_5^{V(\Gamma)}} \subseteq \overline{N_1}$

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- $V_{50} \cap \mathbf{Z}_5^{V(\Gamma)} \subseteq M_2$

We get the inequalities:

$$1520 \leq e_0 + e_1$$
$$1729 \leq 1 + e_2 + e_3.$$

Case 1: $1520 = e_0 + e_1$ and $1729 = e_2 + e_3$.

Case 2: $1521 = e_0 + e_1$ and $1728 = e_2 + e_3$.

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We also know $|Syl_5(K(\Gamma))| = 5^{4975}$, so

$$4975 = e_1 + 2e_2 + 3e_3.$$

Theorem

Let Γ *be an* $srg(3250, 57, 0, 1)$. *Let* e_0 *denote the rank of the Laplacian matrix of* Γ *over a field of characteristic* 5. *Then either*

$$Syl_5(K(\Gamma)) \cong (\mathbf{Z}/5\mathbf{Z})^{1520-e_0} \oplus \left(\mathbf{Z}/5^2\mathbf{Z}\right)^{1732-e_0} \oplus \left(\mathbf{Z}/5^3\mathbf{Z}\right)^{e_0-3}$$

or

$$Syl_5(K(\Gamma)) \cong (\mathbf{Z}/5\mathbf{Z})^{1521-e_0} \oplus \left(\mathbf{Z}/5^2\mathbf{Z}\right)^{1730-e_0} \oplus \left(\mathbf{Z}/5^3\mathbf{Z}\right)^{e_0-2}.$$

What is the 5-rank of L?

Thank you for your attention!