The Critical Group of a Graph

Josh Ducey
James Madison University

Combinatorics Seminar
University of Wisconsin-Madison

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Outline

- Γ a simple graph

- Γ a simple graph
- A adjacency matrix

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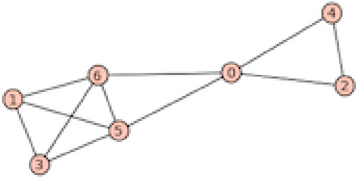
- A adjacency matrix
- $L = D - A$ Laplacian matrix

An example



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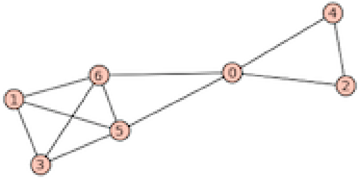

$$A = \begin{bmatrix} & & 1 & & 1 & 1 & 1 \\ & & & 1 & 1 & 1 & \\ 1 & & & & 1 & & \\ & 1 & & & & 1 & 1 \\ 1 & & 1 & & & & \\ 1 & 1 & & 1 & & & 1 \\ 1 & 1 & & 1 & & 1 & \end{bmatrix} \quad L = \begin{bmatrix} 4 & & -1 & & -1 & -1 & -1 \\ & 3 & & -1 & -1 & -1 & \\ -1 & & 2 & & -1 & & \\ & -1 & & 3 & & -1 & -1 \\ -1 & & -1 & & 2 & & \\ -1 & -1 & & -1 & & 4 & -1 \\ -1 & -1 & & -1 & & -1 & 4 \end{bmatrix}$$

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- $\mathrm{Coker}(L) = \mathbb{Z}^k \oplus \mathcal{K}(\Gamma)$
- $\mathcal{K}(\Gamma)$ is the *critical group* (or *sandpile group*, *Jacobian*...)
- $|K(\Gamma)|$ counts number of spanning trees

Known families of critical groups

- trees, $\{0\}$
- n-cycle, Z_n
- complete graph K_n, $(Z_n)^{n-2}$
- wheel graph W_n (n odd), $(Z_{\ell_n})^2$
- line graphs (partial information)
- abelian Cayley graphs (partial information)
- Hypercube graph Q_n (2-part unknown)
- Payley, Peisert graphs
- many others

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- There exist bases of domain and codomain so that M has matrix

$$\begin{bmatrix} s_1 & & & & & \\ & s_2 & & & & \\ & & s_3 & & & \\ & & & \ddots & & \\ & & & & 0 & \\ & & & & & 0 \end{bmatrix},$$

where the s_i are integers with $s_i | s_{i+1}$ for all i.

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where the s_i are integers with $s_i | s_{i+1}$ for all i.
- The s_i are called the invariant factors of M, and

$$\mathrm{Coker}(M) \cong \mathbb{Z}/s_1\mathbb{Z} \oplus \mathbb{Z}/s_2\mathbb{Z} \oplus \cdots$$

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- We label each vertex of the graph with an integer–we call the result a *configuration*

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- Two configurations v_1 and v_2 represent the same element of the critical group if $v_1 - v_2 = Lu$, or

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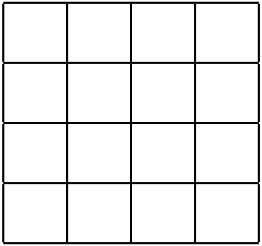
  $$v_1 - Lu = v_2$$

  We can get from v_1 to v_2 by *chip-firing*.
- We may also restrict to configurations with vertices summing to zero.

Example: The rook's graph R_n

- Let R_n be the graph having vertex set the squares of an $n \times n$ grid.



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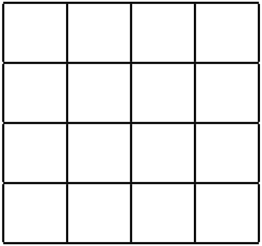

- Two squares are adjacent when they lie in the same row or column.

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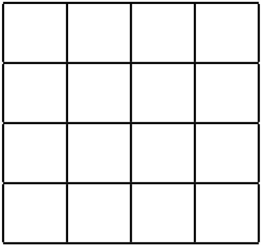

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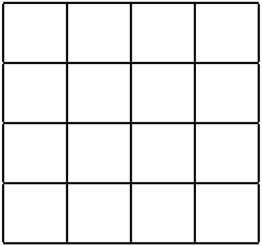

- $v = n^2$
- $k = 2(n-1)$

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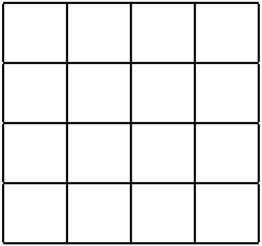

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- $\lambda = n-2$

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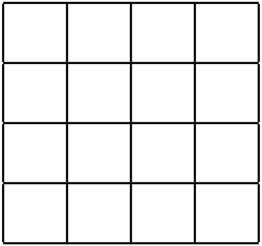

- $v = n^2$
- $k = 2(n-1)$
- $\lambda = n-2$
- $\mu = 2$

Order of the $\mathcal{K}(\Gamma)$

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- Its adjacency spectrum is

$$[-2]^{(n-1)^2}, [n-2]^{2n-2}, [2(n-1)]^1.$$

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$$\begin{aligned} |S(R_n)| &= 2^{(n-1)^2} \cdot (n-2)^{2n-2} \cdot 2(n-1) \\ &= 2^{(n-2)^2} \cdot (2(n-2))^{2n-3} \cdot 2(n-1)(n-2). \end{aligned}$$

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- L has spectrum
$$[2n]^{(n-1)^2}, [n]^{2n-2}, [0]^1.$$
- Matrix tree theorem implies
$$\begin{aligned}|\mathcal{K}(R_n)| &= \frac{1}{n^2} \cdot (2n)^{(n-1)^2} \cdot n^{2n-2} \\ &= (2n)^{(n-2)^2+1} \cdot (2n^2)^{2(n-2)}.\end{aligned}$$

Example: $\mathcal{K}(R_4)$

-8	8		
8	-8		

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-8	8		
8	-8		

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-8	8		
8	-8		

$\Rightarrow$

-2	7	-1	-1
7	-8		
-1			
-1			

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-8	8		
8	-8		

$\Rightarrow$

-2	7	-1	-1
7	-8		
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$\Rightarrow$

-1	1		
7	-7		
-1	1		
-1	1		

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$\Rightarrow$

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7	-8		
-1			
-1			

$\Rightarrow$

-1	1		
7	-7		
-1	1		
-1	1		

$\Rightarrow$

-1			
6	-1	-1	-1
-1			
-1			

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-2	7	-1	-1
7	-8		
-1			
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$\Rightarrow$

-1	1		
7	-7		
-1	1		
-1	1		

$\Rightarrow$

-1			
6	-1	-1	-1
-1			
-1			

$\Rightarrow$

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The order of the critical group, $\mathcal{K}(R_4)$, is $2^{35} = 34359738368$.

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$$\mathcal{K}(R_4) \cong (\mathbb{Z}_8)^5 \oplus (\mathbb{Z}_{32})^4 .$$

Example: $\mathcal{K}(R_4)$

-1	1		
1	-1		

-1	1		
1	-1		

-1		1	
1		-1	

-1		1	
1		-1	

-1	1		

-1		1	

-1			
1			

-1			
1			

-3	1	1	1

Example: $\mathcal{K}(R_4)$

-32	32		

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-32	32		

$\Rightarrow$

-27	2	5	5
	5		
	5		
	5		

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-32	32		

$\Rightarrow$

-27	2	5	5
	5		
	5		
	5		

$\Rightarrow$

3	-3		
-5	5		
-5	5		
-5	5		

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-32	32		

$\Rightarrow$

-27	2	5	5
	5		
	5		
	5		

$\Rightarrow$

3	-3		
-5	5		
-5	5		
-5	5		

$\Rightarrow$

3			
-4	1	1	1
-4	1	1	1
-4	1	1	1

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-32	32		

$\Rightarrow$

-27	2	5	5
	5		
	5		
	5		

$\Rightarrow$

3	-3		
-5	5		
-5	5		
-5	5		

$\Rightarrow$

3			
-4	1	1	1
-4	1	1	1
-4	1	1	1

$\Rightarrow$

Example: $\mathcal{K}(R_4)$

These elements generate the group.

-1	1		
1	-1		

-1	1		
1	-1		

-1		1	
1		-1	

-1		1	
1		-1	

-1	1		

-1		1	

-1			
1			

-1			
1			

-3	1	1	1

Example: $\mathcal{K}(R_4)$

-3	1	1	1

$+$

1	-1		

$+$

1		-1	

$=$

-1			1

Example: $\mathcal{K}(R_4)$

-3	1	1	1

+

1	-1		

+

1		-1	

=

-1			1

-6	1	1	1
1			
1			
1			

+

3	-1	-1	-1

+

1			
-1			

+

1			
-1			

=

-1			
1			

A similar game can be played with the adjacency matrix, and with the graph ${R_n}^c$.

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Theorem (D, Gerhard, Watson)

The critical group and Smith group of R_n and its complement R_n^c are given by the following isomorphisms:

$$\mathcal{K}(R_n) \cong (\mathbb{Z}_{2n})^{(n-2)^2+1} \oplus (\mathbb{Z}_{2n^2})^{2(n-2)}$$
$$S(R_n) \cong (\mathbb{Z}_2)^{(n-2)^2} \oplus \left(\mathbb{Z}_{2(n-2)}\right)^{2n-3} \oplus \mathbb{Z}_{2(n-1)(n-2)}$$
$$\mathcal{K}(R_n^c) \cong \left(\mathbb{Z}_{n(n-2)}\right)^{(n-2)^2-1} \oplus \left(\mathbb{Z}_{n(n-1)(n-2)}\right)^2 \oplus \left(\mathbb{Z}_{n^2(n-1)(n-2)}\right)^{2(n-2)}$$
$$S(R_n^c) \cong \left(\mathbb{Z}_{(n-1)}\right)^{2(n-1)} \oplus \mathbb{Z}_{(n-1)^2}.$$

Example: Kneser graphs

- We let $KG(n, k)$ denote the graph with vertices the size k subsets of an n element set.
- A pair of subsets are adjacent if and only if they are disjoint.

Example: $KG(7,2)$

1						
	2					
		3				
			4			
				5		
					6	
						7

Previous work on $KG(n, 2)$

- A_{comp} - Brouwer and van Eijl (elementary row/col ops, 1993)
- L_{comp} - Berget, et al. (critical groups of line graphs, 2012)
- A - Wilson (SNF bases, 1990)

Example: $KG(7,2)$

1	9					-9
	2				-9	
		3				
			4			
				5		
					6	9
						7

Example: $KG(7, 2)$

1	-1					-9
	2				-9	
		3	1	1	1	1
			4	1	1	1
				5	1	1
					6	10
						7

Example: $KG(7,2)$

1		1	1	1		-9
	2	1	1	1	-9	
		3	2	2	1	1
			4	2	1	1
				5	1	1
					6	
						7

Example: $KG(7,2)$

1						-10
	2	1	1	1	1	
		3	1	1	1	
			4	1	1	
				5	1	
					6	
						7

Example: $KG(7, 2)$

1						
	2					
		3				
			4			
				5		
					6	
						7

Example: $KG(7, 2)$

1	1					-1
	2				-1	
		3				
			4			
				5		
					6	1
						7

This shows that the configuration above represents an element of the critical group with order dividing 9.

Example: $KG(7,2)$

1	1					-1
	2				-1	
		3				
			4			
				5		
					6	1
						7

This shows that the configuration above represents an element of the critical group with order dividing 9.

$\mathbb{Z}_3 \oplus (\mathbb{Z}_9)^7 \oplus \mathbb{Z}_{18} \oplus (\mathbb{Z}_{126})^5$

Outline

Localization

$$L\colon \mathbb{Z}^{V(\Gamma)} \to \mathbb{Z}^{V(\Gamma)}$$

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$$\begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 6 & & \\ & & & 12 & \\ & & & & 0 \end{pmatrix}$$

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$$\begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & 4 & \\ & & & & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 3 & & \\ & & & 3 & \\ & & & & 0 \end{pmatrix}$$

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$$L\colon {\mathbb{Z}_p}^{V(\Gamma)} \to {\mathbb{Z}_p}^{V(\Gamma)}$$

Elementary divisors

- $L\colon \mathbb{Z}_p{}^{V(\Gamma)} \to \mathbb{Z}_p{}^{V(\Gamma)}$
- $M_i = \left\{ x \in \mathbb{Z}_p{}^{V(\Gamma)} \mid Lx \in p^i\, \mathbb{Z}_p{}^{V(\Gamma)} \right\}$
- $N_i = \left\{ p^{-i} Lx \mid x \in M_i \right\}$
- Let e_i denote multiplicity of p^i as elementary divisor of A
- $$\dim_{\mathbb{F}_p} \overline{M_i} = \dim_{\mathbb{F}_p} \overline{\ker(L)} + e_i + e_{i+1} + \cdots$$

  and

  $$\dim_{\mathbb{F}_p} \overline{N_i} = e_0 + e_1 + \cdots + e_i.$$

Example computation

Let $\Gamma = KG(n,2)$. Matrix-tree theorem gives us:

$$|\mathcal{K}(\Gamma)| = \frac{\left[\frac{n(n-3)}{2}\right]^f \left[\frac{(n-4)(n-1)}{2}\right]^g}{\frac{n(n-1)}{2}}$$
$$= \frac{n^{f-1}(n-1)^{g-1}(n-3)^f(n-4)^g}{2^{f+g-1}},$$

where $f = n-1$ and $g = n(n-3)/2$.

Case: $p \neq 2, 3; p \mid n-3$

$$|\mathcal{K}(\Gamma)| = \frac{n^{f-1}(n-1)^{g-1}(n-3)^f(n-4)^g}{2^{f+g-1}}$$

Say $p^a \parallel n-3$. We have

$$\begin{aligned} af = v_p|\mathcal{K}(\Gamma)| = \sum_{i\geq 0} ie_i \geq \sum_{i \geq a} ie_i \geq a \sum_{i\geq a} e_i \\ = a\left(\dim \overline{M_a} - 1\right) \\ \geq a\left((f+1)-1\right) \\ = af. \end{aligned}$$

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Equality throughout. Follows that $e_a = f$, $e_0 = g$, $e_i = 0$ otherwise.

In many cases knowledge of the structure of the permutation module is required. Theory due to G. James (1980s).

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Theorem (D, Hill, Sin '17)

Let $n \geq 5$. Then the critical group of the Kneser graph $\Gamma = KG(n,2)$ has the form

$$K(\Gamma) \cong \begin{cases} \mathbb{Z}_{n-4} \oplus \left(\mathbb{Z}_{\frac{(n-4)(n-1)}{2}}\right)^{\frac{n(n-5)}{2}} \oplus \mathbb{Z}_{\frac{(n-4)(n-1)(n-3)}{4}} \oplus \left(\mathbb{Z}_{\frac{(n-4)(n-1)(n-3)n}{4}}\right)^{n-2} & \text{if } n \text{ is odd,} \\ \mathbb{Z}_{\frac{n-4}{2}} \oplus \left(\mathbb{Z}_{\frac{(n-4)(n-1)}{2}}\right)^{\frac{n(n-5)}{2}} \oplus \mathbb{Z}_{\frac{(n-4)(n-1)(n-3)}{2}} \oplus \left(\mathbb{Z}_{\frac{(n-4)(n-1)(n-3)n}{4}}\right)^{n-2} & \text{if } n \text{ is even.} \end{cases}$$

In recent work with Peter Sin we have computed the elementary divisors for the skew-lines graph in $PG(n, q)$.

2-spaces

- Γ, graph with vertices the 2-dimensional subspaces of an n-dimensional vector space over $\mathbb{F}_q$ ($q = p^t$), adjacent when far apart
- A adjacency matrix, $L = D - A$ Laplacian matrix
- Complement graph denoted Γ'
- A', L'
- Strongly regular with parameters

$$\begin{aligned} v' &= \begin{bmatrix} n \\ 2 \end{bmatrix}_q \\ k' &= q(q+1)\begin{bmatrix} n-2 \\ 1 \end{bmatrix}_q \\ \lambda' &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + q^2 - 2 \\ \mu' &= (q+1)^2. \end{aligned}$$

Skew lines, A

- $A^2 = kI + \lambda A + \mu(J - A - I)$

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- $A\colon \mathbb{Z}_p{}^{V(\Gamma)} \to \mathbb{Z}_p{}^{V(\Gamma)}$
- $\mathbb{Z}_p{}^{V(\Gamma)} = \mathbb{Z}_p \mathbf{1} \oplus Y$

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- $A\colon\ {\mathbb{Z}_\mathrm{p}}^{V(\Gamma)} \to {\mathbb{Z}_\mathrm{p}}^{V(\Gamma)}$
- ${\mathbb{Z}_\mathrm{p}}^{V(\Gamma)} = \mathbb{Z}_\mathrm{p}\,\mathbf{1} \oplus Y$
- $A|_Y[A|_Y - qz_1 I] = q^3 z_2 I$
- A has spectrum

$$q^4 {\begin{bmatrix} n-2 \\ 2 \end{bmatrix}}_q,\ -q^2 {\begin{bmatrix} n-3 \\ 1 \end{bmatrix}}_q,\ q$$

with respective multiplicities $1, {\begin{bmatrix} n \\ 1 \end{bmatrix}}_q - 1, {\begin{bmatrix} n \\ 2 \end{bmatrix}}_q - {\begin{bmatrix} n \\ 1 \end{bmatrix}}_q$.

Skew lines, A

- $$V_{-q^2} \cap \mathbb{Z}_p{}^{V(\Gamma)} \subseteq M_{2t}(A)$$
  $$V_q \cap \mathbb{Z}_p{}^{V(\Gamma)} \subseteq N_t(A).$$

Skew lines, A

- $$V_{-q^2} \cap \mathbb{Z}_p{}^{V(\Gamma)} \subseteq M_{2t}(A)$$
  $$V_q \cap \mathbb{Z}_p{}^{V(\Gamma)} \subseteq N_t(A).$$

- $$\begin{bmatrix} n \\ 1 \end{bmatrix}_q - 1 \leq e_{2t} + \cdots + e_{3t} \quad (1)$$
  $$\begin{bmatrix} n \\ 2 \end{bmatrix}_q - \begin{bmatrix} n \\ 1 \end{bmatrix}_q \leq e_0 + \cdots + e_t. \quad (2)$$

Skew lines, A

- $$V_{-q^2} \cap \mathbb{Z}_p{}^{V(\Gamma)} \subseteq M_{2t}(A)$$
  $$V_q \cap \mathbb{Z}_p{}^{V(\Gamma)} \subseteq N_t(A).$$

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- In general, if $A_{r,s}$ denotes the zero-intersection incidence matrix between r-spaces and s-spaces, then we have that
  $$-A_{r,s} \equiv A_{r,1}A_{1,s} \pmod{p^t}$$

Skew lines, A

Theorem (Brouwer-D-Sin 2011)

Let e_i denote the multiplicity of p^i as a p-adic elementary divisor of $A_{2,1}A_{1,2}$.

1. $e_{4t} = 1$.
2. *For* $i \neq 4t$,

$$e_i = \sum_{\vec{s} \in \Gamma(i)} d(\vec{s}),$$

where

$$\Gamma(i) = \bigcup_{\substack{\alpha+\beta=i \\ 0 \leq \alpha \leq t \\ 0 \leq \beta \leq t}} {}_{\beta}\mathcal{H} \cap \mathcal{H}_{\alpha}.$$

Summation over an empty set is interpreted to result in 0.

- L: same result as for A, but no e_{4t}.
- A': p-part is cyclic of order p^t
- L': no p-part

Notation: The vector space has dimension n over a field of $q = p^t$ elements.

(n, p, t)	matrix	(elem. div. : multiplicity)
(4, 2, 1)	A	$(2:14)$, $(2^2:8)$, $(2^3:6)$, $(2^4:1)$
	L	$(2:14)$, $(2^2:8)$, $(2^3:6)$ $(5:13)$ $(7:19)$
	A'	$(2:1)$ $(3:8)$, $(3^2:14)$
	L'	$(3:8)$, $(3^2:13)$ $(5:13)$ $(7:19)$

Notation: The vector space has dimension n over a field of $q = p^t$ elements.

(n, p, t)	matrix	(elem. div. : multiplicity)
(4, 2, 2)	A	$(2:16), (2^2:220), (2^4:32), (2^5:16), (2^6:36), (2^8:1)$
	L	$(2:16), (2^2:220), (2^4:32), (2^5:16), (2^6:36)$ $(3:1), (3^2:271)$ $(7:271)$ $(17:83)$
	A'	$(2^2:1)$ $(3:84)$ $(5:190), (5^2:84)$
	L'	$(3:271)$ $(5:190), (5^2:83)$ $(7:271)$; $(17:83)$

p' part

- ℓ, a prime different than p
- Structure of $\mathbb{F}_\ell \mathrm{GL}(n, q)$-permutation module on 2-spaces is able to be understood. (few composition factors)
- We rely heavily on work of G. James
- $\mathbb{F}_\ell^{V(\Gamma)}$ has descending filtration with subquotients as Specht modules
- Straightforward arithmetic conditions determine the composition factors and multiplicities of the Specht modules

Final Example.

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L has spectrum $[50]^{1729}, [65]^{1520}, [0]^1$, so

$$
\begin{aligned}
|\mathcal{K}(\Gamma)| &= \frac{1}{3250} \cdot 50^{1729} \cdot 65^{1520} \\
&= 2^{1728} \cdot 5^{4975} \cdot 13^{1519}.
\end{aligned}
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$$L^2 + (\lambda - \mu - 2k)L = (k - k^2 + \lambda k - \mu - \mu k)I + \mu J$$
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$$K(\Gamma) \cong (\mathbb{Z}/2\mathbb{Z})^{1728} \oplus (\mathbb{Z}/13\mathbb{Z})^{1519} \oplus (\mathbb{Z}/5\mathbb{Z})^{e_1} \oplus \left(\mathbb{Z}/5^2\mathbb{Z}\right)^{e_2} \oplus \left(\mathbb{Z}/5^3\mathbb{Z}\right)^{e_3}$$

Consider the inclusions of the eigenspaces of L in M_i, N_j:

- $V_{65} \cap \mathbb{Z}_5^{V(\Gamma)} \subseteq N_1$, and so $\overline{V_{65} \cap \mathbb{Z}_5^{V(\Gamma)}} \subseteq \overline{N_1}$

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- $V_{50} \cap \mathbb{Z}_5^{V(\Gamma)} \subseteq M_2$

We get the inequalities:

$$\begin{aligned} 1520 &\leq e_0 + e_1 \\ 1729 &\leq 1 + e_2 + e_3. \end{aligned}$$

Case 1: $1520 = e_0 + e_1$ and $1729 = e_2 + e_3$.

Case 2: $1521 = e_0 + e_1$ and $1728 = e_2 + e_3$.

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We also know $|Syl_5(K(\Gamma))| = 5^{4975}$, so

$$4975 = e_1 + 2e_2 + 3e_3.$$

Theorem

Let Γ *be an* $srg(3250, 57, 0, 1)$*. Let* e_0 *denote the rank of the Laplacian matrix of* Γ *over a field of characteristic* 5*. Then either*

$$Syl_5(K(\Gamma)) \cong (\mathbb{Z}/5\mathbb{Z})^{1520-e_0} \oplus \left(\mathbb{Z}/5^2\mathbb{Z}\right)^{1732-e_0} \oplus \left(\mathbb{Z}/5^3\mathbb{Z}\right)^{e_0-3}$$

or

$$Syl_5(K(\Gamma)) \cong (\mathbb{Z}/5\mathbb{Z})^{1521-e_0} \oplus \left(\mathbb{Z}/5^2\mathbb{Z}\right)^{1730-e_0} \oplus \left(\mathbb{Z}/5^3\mathbb{Z}\right)^{e_0-2}.$$

Some problems

- hypercube (2-part)
- SNF of other matrices attached to graphs
- DRGs
- see Stanley's survey article

Thank you for your attention!