Finite abelian groups attached to graphs.

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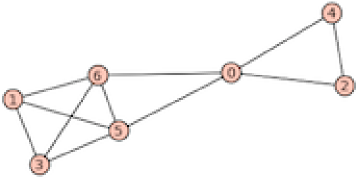
Outline

1. The Smith and critical groups
2. Modules and Smith Normal Form
3. Skew Lines

An example



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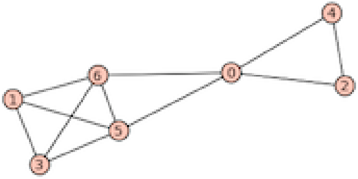

$$A = \begin{bmatrix} & & 1 & & 1 & 1 & 1 \\ & & & 1 & 1 & 1 & \\ 1 & & & & 1 & & \\ & 1 & & & & 1 & 1 \\ 1 & & 1 & & & & \\ 1 & 1 & & 1 & & & 1 \\ 1 & 1 & & 1 & & 1 & \end{bmatrix} \qquad L = \begin{bmatrix} 4 & & -1 & & -1 & -1 & -1 \\ & 3 & & -1 & -1 & -1 & \\ -1 & & 2 & & -1 & & \\ & -1 & & 3 & & -1 & -1 \\ -1 & & -1 & & 2 & & \\ -1 & -1 & & -1 & & 4 & -1 \\ -1 & -1 & & -1 & & -1 & 4 \end{bmatrix}$$

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- The critical group also appears in many contexts under many different names (Picard group, Jacobian group, sandpile group, chip-firing, etc.)

Known critical groups

- trees
- n-cycle
- wheel graphs, odd number of vertices
- complete graphs
- conference graphs on square-free number of vertices
- Payley graphs
- complete multipartite graphs
- abelian Cayley graphs (partial information)
- line graphs (partial information)

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  where the s_i are integers with $s_i | s_{i+1}$ for all i.
- The s_i are called the invariant factors of M, and

$$\mathrm{Coker}(M) \cong \mathbb{Z}/s_1\mathbb{Z} \oplus \mathbb{Z}/s_2\mathbb{Z} \oplus \cdots$$

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- Γ is an $SRG(v, k, \lambda, \mu)$ with

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- Our graph is unique among these with minimal 2-rank of 6.

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- For a field F of characteristic $\ell \neq p$, the FG-module structure of $F^{\mathcal{L}_2}$ is known.
- The modules $\overline{M_i}$ can be identified and their dimensions computed.

Cases

There are several cases.

- case 1: $\ell \neq 2$, $\ell \nmid (q^2+1)$
  - 1a: $\ell \mid (q-1)$, $\ell \nmid (q^2+q+1)$
  - 1b: $\ell \nmid (q-1)$, $\ell \mid (q^2+q+1)$
  - 1c: $\ell \mid (q-1)$, $\ell \mid (q^2+q+1)$ (so $\ell = 3$)
- case 2: $\ell \neq 2$, $\ell \mid (q^2+1)$
- case 3: $\ell = 2$

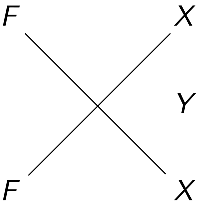
Case 1a

$$F \quad \oplus \quad X \quad \oplus \quad Y$$

Cases 1b, 1c, 2

$$\begin{array}{ccc} F & & \\ X & \oplus & Y \\ F & & \end{array}$$

Case 3 (ℓ=2)



Result

Theorem

- *case 1a: Suppose* $\ell^a \mid\mid (q-1)$. *Then*

$$e_0 = q^3 + q^2 + q, e_a = q^4 + q^2, e_i = 0 \textit{ otherwise}$$

- *case 1b: Suppose* $\ell^a \mid\mid (q^2+q+1)$. *Then*

$$e_0 = q^3 + q^2 + q, e_a = q^4 + q^2 - 1, e_i = 0 \textit{ otherwise}$$

- *case 1c* ($\ell = 3$)*: Suppose* $\ell^a \mid\mid (q-1)$. *Then*

$$e_0 = q^3 + q^2 + q, e_a = 1, e_{a+1} = q^4 + q^2 - 1, e_i = 0 \textit{ otherwise}$$

- *case 2: Suppose* $\ell^a \mid\mid (q^2+1)$. *Then*

$$e_0 = q^4 + q^2 + 1, e_a = q^3 + q^2 + q - 1, e_i = 0 \textit{ otherwise}$$

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Theorem

case 3 ($\ell = 2$): Suppose $\ell^a \,||\, (q-1)(q^2+q+1)$. Then

$$\begin{aligned} e_0 &= q^3+q^2+q-1, \\ e_a &= q^4-q^3-q^2+1, \\ e_{a+1} &= q^3+q^2+q, \\ e_i &= \ \textit{otherwise} \end{aligned}$$

Example

For skew lines in $PG(3,7)$, the Laplacian SNF is:

1^{231}	7^{168}	42^{1}	2394^{2050}	16758^{1}	837900^{167}	5865300^{231}	0^{1}
		3	3^2	3^2	3^2	3^2	

Thank you for your attention!