Abelian groups associated to strongly regular graphs.

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SECANT 3
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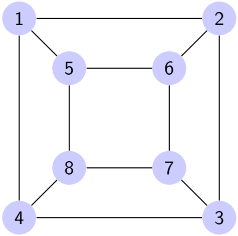
Outline

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$



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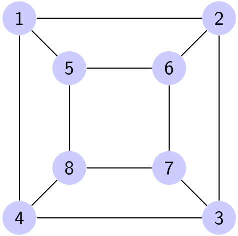

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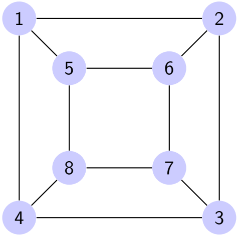

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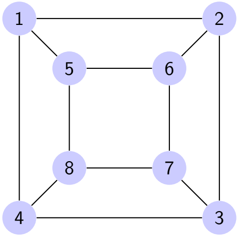
- Also popular, the Laplacian matrix $L = D - A$, where D is diagonal matrix of vertex degrees.

$$L = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$



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- Could also consider $\mathrm{Coker}(A + bI + cJ)$

Examples of known sandpile groups

- n-cycle
- complete graphs
- n-cube (2-part still unsolved)
- many families of strongly regular graphs
- good source of problems to work on with motivated undergraduates

There are many ways to think about and compute these groups.

- Smith normal form of the matrix
- chip-firing games on the graphs
- matroids, representation theory, ...

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- any two adjacent vertices have exactly λ common neighbors
- any two distinct, non-adjacent vertices have exactly μ common neighbors.

- It follows: $A^2 = kI + \lambda A + \mu(J - A - I)$, and A has exactly two distinct eigenvalues besides the degree k.
- Paley, Kneser on 2-subsets and projective lines, rook, polar, latin square
- These are cool graphs, and so the existence question for particular parameter sets is quite interesting.

Andries Brouwer's website

Prev Up Next

	v	k	λ	μ	r^f	s^g	comments
!	153	32	16	4	14^{17}	-2^{135}	Triangular graph T(18)
		120	91	105	1^{135}	-15^{17}	pg(8,14,7)
?	153	56	19	21	5^{84}	-7^{68}	pg(8,6,3)?
		96	60	60	6^{68}	-6^{84}	
?	153	76	37	38	5.685^{76}	-6.685^{76}	2-graph*?
?	154	48	12	16	4^{98}	-8^{55}	pg(6,7,2)?
		105	72	70	7^{55}	-5^{98}	
-	154	51	8	21	2^{132}	-15^{21}	Krein2
		102	71	60	14^{21}	-3^{132}	Krein1
?	154	72	26	40	2^{132}	-16^{21}	
		81	48	36	15^{21}	-3^{132}	
+	155	42	17	9	11^{30}	-3^{124}	S(2,3,31); lines in PG(4,2)
		112	78	88	2^{124}	-12^{30}	
+	156	30	4	6	4^{90}	-6^{65}	O(5,5) Sp(4,5); GQ(5,5)
		125	100	100	5^{65}	-5^{90}	
+	157	78	38	39	5.765^{78}	-6.765^{78}	Paley(157); 2-graph*
?	160	54	18	18	6^{75}	-6^{84}	pg(9,5,3) does not exist (no 2-graph* for line graph)
		105	68	70	5^{84}	-7^{75}	
-	161	80	39	40	5.844^{80}	-6.844^{80}	Conf
?	162	21	0	3	3^{105}	-6^{56}	

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2019 REU at James Madison University

Work with D. Duncan, W. Engelbrecht, J. Madan, E. Piato, C. Shatford, A. Vichitbandha. Suppose Γ is an srg(v,k,λ,μ) with nonzero Laplacian eigenvalues r^f, s^g.

- $p \nmid r, p^a \parallel s, p^\gamma \parallel v \implies K_p(\Gamma) \cong \mathbb{Z}/p^{a-\gamma}\mathbb{Z} \oplus (\mathbb{Z}/p^a\mathbb{Z})^{g-1}$.
- $p \parallel r, p \parallel s, p^\gamma \parallel v \implies K_p(\Gamma) \cong (\mathbb{Z}/p\mathbb{Z})^{f+g+\gamma-2e_0} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{e_0-\gamma}$.
- $p \parallel r, p^2 \parallel s, p^\gamma \parallel v \implies$

$$K_p(\Gamma) \cong (\mathbb{Z}/p\mathbb{Z})^{f-e_0} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{g+\gamma-e_0} \oplus (\mathbb{Z}/p^3\mathbb{Z})^{e_0-\gamma}$$

or

$$K_p(\Gamma) \cong (\mathbb{Z}/p\mathbb{Z})^{f+1-e_0} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{g+\gamma-2-e_0} \oplus (\mathbb{Z}/p^3\mathbb{Z})^{e_0-\gamma+1}.$$

Some consequences

- If Γ is an $\mathrm{srg}(99, 14, 1, 2)$, then

$$K(\Gamma) \cong (\mathbb{Z}/11\mathbb{Z})^{53} \oplus (\mathbb{Z}/2\mathbb{Z})^{44} \oplus (\mathbb{Z}/9\mathbb{Z})^{43}.$$

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- If Γ is an srg$(3250, 57, 0, 1)$, then

$$K_5(\Gamma) \cong (\mathbb{Z}/5\mathbb{Z})^{1520-e_0} \oplus (\mathbb{Z}/25\mathbb{Z})^{1732-e_0} \oplus (\mathbb{Z}/125\mathbb{Z})^{e_0-3}$$

or

$$K_5(\Gamma) \cong (\mathbb{Z}/5\mathbb{Z})^{1521-e_0} \oplus (\mathbb{Z}/25\mathbb{Z})^{1730-e_0} \oplus (\mathbb{Z}/125\mathbb{Z})^{e_0-2}.$$

Some consequences

- If Γ is an srg$(28, 9, 0, 4)$, then the 7-rank of L must be 22, and so the kernel of L modulo 7 must have dimension 6. But it is not difficult to cook up more than 6 independent vectors in this kernel.

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- If Γ is an srg$(28, 9, 0, 4)$, then the 7-rank of L must be 22, and so the kernel of L modulo 7 must have dimension 6. But it is not difficult to cook up more than 6 independent vectors in this kernel.
- By the way, the smallest open parameter set is $(69, 20, 7, 5)$.

- It turns out that these results for the Laplacian can be extended without much difficulty to the adjacency as well, with care, and also to matrices of the form $A + bI + cJ$.
- Generally we need these matrices to be nonsingular (but not always), and in general exactly one invariant factor will be uncertain.

But, for instance, can deduce

- If Γ is an srg$(99, 14, 1, 2)$, then

$$S(\Gamma) \cong \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z})^{44} \oplus (\mathbb{Z}/3\mathbb{Z})^{54} \oplus \mathbb{Z}/7\mathbb{Z}.$$

- If Γ is an srg$(3250, 57, 0, 1)$, then

$$S(\Gamma) \cong \mathbb{Z}/4\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z})^{45} \oplus (\mathbb{Z}/5\mathbb{Z})^{24}.$$

Interlacing

- Interlacing of adjacency eigenvalues of a graph with those of an induced subgraph is a standard technique from spectral graph theory.
- New to me is a result of R. C. Thompson from 1978 that describes an interlacing result of invariant factors of an integer matrix with those of any submatrix.
- From this result we can deduce:

Theorem

Let $N = A + bI + cJ$ describe a graph Γ. Suppose that a cyclic decomposition of the Sylow p-subgroup of $\mathrm{Coker}(N)$ contains exactly e_i summands $\mathbb{Z}/p^i\mathbb{Z}$. Remove k vertices of Γ to get an induced subgraph H, and delete those corresponding k rows and columns of N to get a matrix N_H. Then the corresponding Sylow p-subgroup of $\mathrm{Coker}(N_H)$ must have at least $e_i - 2k$ summands $\mathbb{Z}/p^i\mathbb{Z}$.

Interlacing example

- Let Γ be an $\mathrm{srg}(640, 243, 66, 108)$. Then

  $$S_3(\Gamma) \cong (\mathbb{Z}/3\mathbb{Z})^{594-e_0} \oplus (\mathbb{Z}/9\mathbb{Z})^{45-e_0} \oplus (\mathbb{Z}/27\mathbb{Z})^{e_0} \oplus \mathbb{Z}/243\mathbb{Z}.$$

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- Let Γ_2 be a second subconstituent of Γ; that is, the subgraph induced by all vertices not adjacent to some fixed vertex.
- This is obtained by removing 244 vertices, and so the number of occurrences of $\mathbb{Z}/3\mathbb{Z}$ as a summand of $S_3(\Gamma_2)$ is *at least*

  $$(594 - e_0) - 2(244) = 106 - e_0 \geq 61,$$

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- Further analysis could be done to Γ_2, which also happens to be strongly regular.

Thank you for your attention!