A representation-theoretic approach to understanding some graph matrices.

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James Madison University

AMS Spring Central Sectional Meeting
Recent Trends in Graph Theory

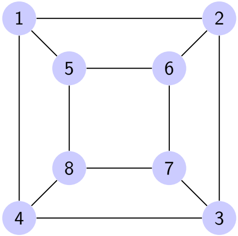
April 16, 2023

In this talk I will be describing joint work with Colby Sherwood.

Outline

1. Integer invariants of graphs
2. Representations of $\mathfrak{S}_n$
3. Hypercube graph

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$



- Γ, a finite simple graph with adjacency matrix A.

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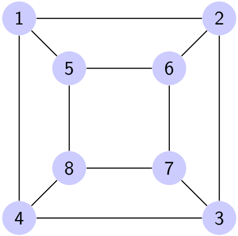
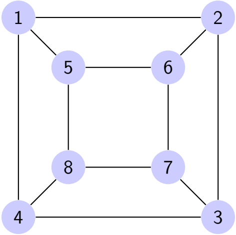


- Γ, a finite simple graph with adjacency matrix A.
- Various other matrices can be used, for example, the Laplacian

$$L = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$



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$$M\colon \mathbb{Z}^n \to \mathbb{Z}^n$$

The cokernel of this map $\mathbb{Z}^n / \operatorname{Im}(M)$ is a finitely generated abelian group:

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unimodular

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Over $\mathbb{R}$:

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ & & & 1 & & & \\ & & & & 1 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

The cokernel of, say, A can change depending on which ring the entries of the matrix come from.
Over $\mathbb{Z}$:

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 2 & & & & \\ & & & 6 & & & \\ & & & & 12 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

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Over $\mathbb{Z}_{(2)}$:

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 2 & & & & \\ & & & 2 & & & \\ & & & & 4 & & \\ & & & & & 0 & \\ & & & & & & 0 \end{pmatrix}$$

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sandpile group

How to find the cokernel?

We can find each p-primary component (Sylow subgroup) of the cokernel separately. Let f_i denote the number of copies of $\mathbb{Z}/p^i\mathbb{Z}$ in the p-primary component.

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- $f_i = \dim_p \overline{M_i}/\overline{M_{i+1}} = \dim_p \overline{N_i}/\overline{N_{i-1}}$

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If the vertices of your graph are subsets, and the action of the symmetric group $\mathfrak{S}_n$ preserves adjacency, then both the domain and codomain of L are permutation modules.

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A great deal of information about their submodule structure comes from theory of G. James.

$$t = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}$$

$$\{t\} = \begin{array}{cccc}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array} = \begin{array}{cccc}\hline 2 & 1 & 3 & 4 \\ \hline 6 & 5 \\ \cline{1-2}\end{array} = \begin{array}{cccc}\hline 1 & 2 & 4 & 3 \\ \hline 5 & 6 \\ \cline{1-2}\end{array}$$

$$e_t^0 = \{t\}$$

$$e_t^1 = \begin{array}{cccc}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array} - \begin{array}{cccc}\hline 5 & 2 & 3 & 4 \\ \hline 1 & 6 \\ \cline{1-2}\end{array}$$

$$e_t^2 = \begin{array}{cccc}\hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2}\end{array} - \begin{array}{cccc}\hline 5 & 2 & 3 & 4 \\ \hline 1 & 6 \\ \cline{1-2}\end{array} - \begin{array}{cccc}\hline 1 & 6 & 3 & 4 \\ \hline 5 & 2 \\ \cline{1-2}\end{array} + \begin{array}{cccc}\hline 5 & 6 & 3 & 4 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}$$

$$M^{(n-i,i)}$$
$$\cup$$
$$S^{(n-i,1)(n-i,i)}$$
$$\cup$$
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$$\cup$$
$$\vdots$$
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\vdots \\
\cup \\
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\end{array}
\quad \sim \quad
\begin{array}{c}
S^{0} \\
\hline
S^{1} \\
\hline
S^{2} \\
\hline
\vdots \\
\hline
S^{i}
\end{array}
$$

$$M^{(n-i,i)} \supseteq S^{(n-i,1)(n-i,i)} \supseteq S^{(n-i,2)(n-i,i)} \supseteq \cdots \supseteq S^{(n-i,i)(n-i,i)} = S^i$$

$$\sim \quad \frac{S^0}{\dfrac{S^1}{\dfrac{S^2}{\dfrac{\vdots}{S^i}}}}$$

$$S^j \sim \frac{D^j}{D^k}$$

$k<j$, multiplicity 0 or 1

It follows that for an $F\mathfrak{S}_n$-submodule U of the codomain of L, we get a decending filtration

$$P^k = U \cap S^{(n-i,k)(n-i,i)}, \quad k \geq 0,$$

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$$P^k = U \cap S^{(n-i,k)(n-i,i)}, \quad k \geq 0,$$

Each subquotient P^k/P^{k+1} is isomorphic to a submodule of S^k.

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The n-cube graph Q_n

Vertices:

$$\{(a_1, a_2, \cdots, a_n) \mid a_i = 0 \text{ or } 1\}$$

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Edges: two vertices are adjacent if they differ in exactly one position.

Clearly the vertices may be viewed as subsets of an n-element set.

Work of Bai, Jacobson-Niedermeier-Reiner, and others show that the Laplacian integer invariants (i.e., sandpile group) can be understood by the p-primary components, for all primes except $p = 2$.

Sandpile group of Q_n: $\kappa(Q_n)$

For $p \neq 2$,

$$Syl_p(\kappa(Q_n)) \cong Syl_p\left(\oplus_{j=1}^{n} (\mathbb{Z}/2j\mathbb{Z})^{\binom{n}{j}}\right)$$

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The 2-part of the adjacency cokernel was found by work of Chandler-Sin-Xiang. Still not even a conjecture for $Syl_2(\kappa(Q_n))$.

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On the critical group of the n-cube

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Submitted by R. Guralnick

Abstract

Reiner proposed two conjectures about the structure of the critical group of the n-cube Q_n. In this paper we confirm them. Furthermore we describe its p-primary structure for all odd primes p. The results are generalized to Cartesian product of complete graphs $K_{n_1} \times \cdots \times K_{n_k}$ by Jacobson, Niedermaier and Reiner.

Keywords: n-Cube; Critical group; Sandpile group; Laplacian matrix; Smith normal form; Sylow p-group

n	$\mathrm{Syl}_2 K(Q_n)$
2	$\mathbb{Z}_4$
3	$\mathbb{Z}_2\ \mathbb{Z}_8^2$
4	$\mathbb{Z}_2^2\ \mathbb{Z}_8^4\ \mathbb{Z}_{32}$
5	$\mathbb{Z}_2^6\ \mathbb{Z}_8^4\ \mathbb{Z}_{16}\ \mathbb{Z}_{64}^4$
6	$\mathbb{Z}_2^{12}\ \mathbb{Z}_4^4\ \mathbb{Z}_8\ \mathbb{Z}_{32}^4\ \mathbb{Z}_{64}^{10}$
7	$\mathbb{Z}_2^{28}\ \mathbb{Z}_4\ \mathbb{Z}_{16}^8\ \mathbb{Z}_{32}^6\ \mathbb{Z}_{64}^{14}\ \mathbb{Z}_{128}^6$
8	$\mathbb{Z}_2^{56}\ \mathbb{Z}_4^2\ \mathbb{Z}_{16}^{16}\ \mathbb{Z}_{32}^{12}\ \mathbb{Z}_{64}^{28}\ \mathbb{Z}_{128}^{12}\ \mathbb{Z}_{1024}$
9	$\mathbb{Z}_2^{120}\ \mathbb{Z}_4^{10}\ \mathbb{Z}_{16}^{16}\ \mathbb{Z}_{32}^{26}\ \mathbb{Z}_{64}^{48}\ \mathbb{Z}_{128}^{26}\ \mathbb{Z}_{512}\ \mathbb{Z}_{2048}^8$
10	$\mathbb{Z}_2^{240}\ \mathbb{Z}_4^{36}\ \mathbb{Z}_8^{26}\ \mathbb{Z}_{32}^{16}\ \mathbb{Z}_{64}^{148}\ \mathbb{Z}_{256}\ \mathbb{Z}_{1024}^{26}\ \mathbb{Z}_{2048}^{18}$
11	$\mathbb{Z}_2^{496}\ \mathbb{Z}_4^{66}\ \mathbb{Z}_8^{32}\ \mathbb{Z}_{16}^{100}\ \mathbb{Z}_{64}^{164}\ \mathbb{Z}_{128}\ \mathbb{Z}_{512}^{100}\ \mathbb{Z}_{2048}^{64}$

Example: $n = 3$

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$$\frac{S^0 \oplus S^0 \oplus S^0 \oplus S^0}{S^1 \oplus S^1}$$

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Idea:

- Can show that for an $(n-i, i)$-tableau t, $L(e_t^j)$ represents, in the j-th row of the picture,

$$(0, 0, \cdots, (i-j)e_{s'}^j, -2ie_{t'}^j, 0, \cdots, 0).$$

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- The image of the sum of all the 2-subsets under L shows that the sum of all the 1-subsets lies in N_1. This generates an additional copy of S^0 in $\overline{N_1}$.

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- Modulo 2, this is zero unless $i-j$ is odd. So we get at least every other copy of S^j in the j-th row in the image of L.
- The image of the sum of all the 2-subsets under L shows that the sum of all the 1-subsets lies in N_1. This generates an additional copy of S^0 in $\overline{N_1}$.
- For a 2-subset $\{t\}$ containing a 1-subset $\{t'\}$,

$$L(e_{t'}^1 + 2e_t^1) = -8e_t^1.$$

Shows the remaining copy of S^1 lies in $\overline{N_3}$

Thank you for your attention!