Math 238 Extra Credit Project Name:

Due on 12/5/2012 at the beginning of class. You will be graded both on how far you make it through the project, and how clearly you express your ideas and reasoning.

This project will introduce you to the interesting topic of linear algebra over a finite field. Instead of the real or complex numbers, in this project our scalars will be the field of two elements:

$$\mathbb{F} = \{0, 1\}$$

where the addition and multiplication is by definition

$$\begin{aligned} 0 \cdot 0 &= 0 \\ 0 \cdot 1 &= 0 \\ 1 \cdot 1 &= 1 \\ 0 + 0 &= 0 \\ 0 + 1 &= 1 \\ 1 + 1 &= 0. \end{aligned}$$

Perhaps the last equation above looks funny to you–this is sometimes called addition *modulo 2*. With these rules all computations with 0 and 1 stay inside this finite field. All of the usual laws of arithmetic still hold, like associative and commutative addition and multiplication, and the distributive laws still work. Just remember that $1 + 1 = 0$, and that any scalar appearing anywhere in this project should be either a zero or a one.

You will be using linear algebra to analyze the classic game "Lights Out." If you just google "lights out puzzle" or something similar you can play the game online. A pretty cool website is

`http://www.ueda.info.waseda.ac.jp/~n-kato/lightsout/index.html`

but you may find a better one by searching around.

Lights Out is a one player puzzle where you are given a rectangular grid of squares with some of the squares "on" and some of the squares "off" (or some red and some blue, etc.). When you press a square, that square and its vertical and horizontal neighbors will change their states (each will go from off to on, or from on to off). It might be a good idea to play with the puzzle a little bit at this point until you understand how things work. The goal of the game is to turn all of the states to "off", hence the name.

Before going on I will remark that although we will use linear algebra over the finite field $\mathbb{F}$ to completely understand this puzzle, there is an enormous amount of serious applications of finite fields in mathematics, science, and engineering, ranging from finite geometry to error-correcting codes. These structures are also naturally beautiful. You may learn about some of these in a later course. If you find yourself having fun, or you are interested in these applications or even the idea of doing research in mathematics, please feel free to talk to me about this stuff!

The classic game of Lights Out is played on a 5×5 board, but we will first consider the 3×3 game. As discussed in class, a 3×3 game board with some configuration of lights on and off can be viewed as a matrix of zeros and ones (say zero corresponds to "off" and one corresponds to "on"). In other words, each possible game board is just an element of $Mat_{3\times 3}(\mathbb{F})$.

- One of the interesting things about working over finite fields is that all vector spaces are actually finite *sets*! Count the number of matrices in $Mat_{3\times 3}(\mathbb{F})$. (In other words, count the number of possible starting configurations for the lights out game on a 3×3 board.)
- Write down your favorite matrix A in $Mat_{3\times 3}(\mathbb{F})$, and view it as a configuration of the Lights Out game. Pressing the top–left entry will change A into a different matrix B. Write down this new matrix.

Now let us begin to exploit the fact that we are working over $\mathbb{F}$ instead of over $\mathbb{R}$. Your matrix A can be changed into B using matrix addition. Let $W_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. You should have that $A + W_1 = B$.

- Pressing the top-left button on a game board is the same as adding the matrix W_1. Find matrices $W_2, \cdots, W_9$ that do this for the other buttons.

Let C denote an arbitrary Lights Out game configuration; that is, C is any matrix in $Mat_{3\times 3}(\mathbb{F})$. Notice that starting with C and playing the Lights Out game (i.e., pushing buttons) is the same as adding to C some of the matrices W_i above. Since matrix addition is commutative, this is one way to see that it doesn't matter in what order we press the buttons while playing the game. Also, since $W_i + W_i$ is the zero matrix, pressing the same button twice doesn't do anything (this should be obvious anyway), and so we never need to press a button more than once.

- Explain why it is possible to win the Lights Out game for a game board C precisely when it is possible to write C as a linear combination of the matrices $W_1, \cdots, W_9$.

It turns out actually that for most of the starting configurations of the classic 5×5 Lights Out it is not possible to turn off all of the lights. However, when you play the game on a 3×3 board it is indeed always possible to win. You are going to show this.

- Fix your favorite basis α of $Mat_{3\times3}(\mathbb{F})$ and write down the coordinate vectors of the W_i with respect to this basis.

Now that we are working with column vectors we are ready to do some computations. Take the coordinate vectors of the W_i that you just got, and put them as the rows of a 9×9 matrix. Call this large matrix M.

- If the matrix M has rank 9 then it is always possible to win any game of Lights Out on a 3×3 board. Convince yourself that this statement is true, and then carefully convince me.
- Now roll up your sleeves and actually show that M has rank 9. I suggest row reducing to echelon form, but you can do it however you like. It is a large matrix, but over the field of two elements row operations are lightning fast! (Besides swapping, the only nontrivial row operation is to add one row to another...)

Some more work to do:

- Show that when playing Lights Out on a 2×3 board it is *not* always possible to win. This should go much faster since there are only six buttons this time (hence M will be 6×6).

- If you have a starting configuration of Lights Out on a 2×3 board, and it is possible to win, then how many distinct solutions are there?
  Hint: Computing an actual solution to the Lights Out game is the same as solving the matrix system $M^T\vec{x} = \vec{b}$, where $\vec{b}$ is the coordinate vector of the starting configuration. You get multiple solutions to a system of linear equations when there are "free variables", and you should be able to know how many free variables there will be just by looking at the rank of M. Since you are working over $\mathbb{F}$ there are only finitely many choices for what the free variables can be, resulting in finitely many solutions to

the system. Recall that this never happens over $\mathbb{R}$; over the real numbers there is either no solution, one solution, or infinitely many.

- Make up your own Lights Out–type game. You decide the shape of the board, and what happens when you press each button. Doesn't even have to be rectangular, if you want to be creative. Will there always be a solution to your puzzle?

- **(Optional)** Market and sell your idea; or if you are nice, create a free smartphone app so that everyone can play your game.