Homework Assignment ~~1~~ 2

Due Wednesday 9/10/2014 at start of class.

Assigned:
Section 2.3: 3, 6, 11, 17, 18, 27
Section 2.4: 4, 5, 6, 7, 22, 24, 32
Section 2.5: 6, 15, 18, 25, 29, 32, 44

Collected:

(1) Solve the linear system. If there is a unique solution, find it. If there is no solution, show why. If there are many solutions, try to describe them all (if this is too difficult, give at least 3 distinct solutions).

(a)

$$\begin{aligned} x - y + z &= 0 \\ 2x - 3y + 4z &= -2 \\ -2x - y + z &= 7 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 1 \\ x_1 + x_2 + x_4 &= 2 \\ x_1 + 2x_2 - 4x_3 &= 1 \\ 2x_1 + x_2 + 2x_3 + 5x_4 &= 1 \end{aligned}$$

(c)

$$\begin{aligned} 2x + 3y - z &= 3 \\ -x - y + 3z &= 0 \\ x + 2y + 2z &= 3 \\ y + 5z &= 3 \end{aligned}$$

Give a quick verbal description of the geometric picture behind each system of equations (i.e., the row picture). Was the solution to parts (b) and (c) expected or unexpected?

(2) Find a quadratic function

$$y = ax^2 + bx + c$$

that passes through the points $(1, 4)$, $(-1, -2)$, $(5, 64)$.

(3) Which elimination matrices E_{21} and E_{31} produce zeros in the $(2,1)$ and $(3,1)$ positions of $E_{21}A$ and $E_{31}A$?

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix}$$

Find the single matrix E that produces both zeros at once. Multiply EA to check that it works.

(4) Describe all vectors that are perpendicular to $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$.

(5) Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}.$$

Show that A^{-1} exists (by finding it!) and then use A^{-1} to quickly solve the systems:

(a) $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(b) $A\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(c) $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

① (a)

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -3 & 4 & -2 \\ -2 & -1 & 1 & 7 \end{array}\right] \xrightarrow[2R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & -3 & 3 & 7 \end{array}\right]$$

$$\xrightarrow{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -3 & 13 \end{array}\right]$$

$-3z = 13$

$\boxed{z = \frac{13}{-3}}$

$-y + 2z = -2$

$-y + 2 \cdot \frac{13}{-3} = -2$

$-y = \frac{3 \cdot -2}{3} + \frac{26}{3}$

$-y = \frac{20}{3}$

$\boxed{y = \frac{-20}{3}}$

$x - y + z = 0$

$x + \frac{20}{3} - \frac{13}{3} = 0$

$x + \frac{7}{3} = 0$

$\boxed{x = \frac{-7}{3}}$

(b)

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 2 \\ 1 & 2 & 0 & -4 & 1 \\ 2 & 1 & 2 & 5 & 1 \end{array}\right] \xrightarrow[-2 \cdot R_1 + R_4]{-1 \cdot R_1 + R_2,\ -1 \cdot R_1 + R_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -6 & 0 \\ 0 & -1 & 4 & 1 & -1 \end{array}\right]$$

these were switched in original problem.

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -6 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & -1 & 4 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{1 \cdot R_2 + R_4} \left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -6 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 5 & -5 & -1 \end{array}\right]$$

$$\xrightarrow{-5R_3+R_4}\left[\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & -6 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right]$$

Last row says that $0=-6$.

This is impossible, so system has $\boxed{\text{no solution.}}$

(c)
$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 3 \\ -1 & -1 & 3 & 0 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 5 & 3 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ -1 & -1 & 3 & 0 \\ 2 & 3 & -1 & 3 \\ 0 & 1 & 5 & 3 \end{array}\right]$$

$$\xrightarrow[-2\cdot R_1+R_3]{1\cdot R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & 5 & 3 \\ 0 & -1 & -5 & -3 \\ 0 & 1 & 5 & 3 \end{array}\right]$$

$$\xrightarrow[-1\cdot R_2+R_4]{1\cdot R_2+R_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & 5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Choosing values for z will generate solutions!

$y+5z=3$

$\boxed{y=-5z+3}$

$x+2y+2z=3$

$x+2(-5z+3)+2z=3$

$x-10z+6+2z=3$

$\boxed{x=8z-3}$

For system (a), we are seeking the intersection of three planes in $\mathbb{R}^3$. This turns out to be a point, as expected.

I expected the four hyperplanes given by the equations in (b) to meet in a point, but this was not the case.

The row picture for system (c) is the intersection of four planes in $\mathbb{R}^3$. I expect this intersection to be empty, but it turns out that they all meet in a line.

(2) Suppose $y = ax^2 + bx + c$ passes through $(1,4)$, $(-1,-2)$, and $(5,64)$.

Then

$$\begin{aligned} a + b + c &= 4 \\ a - b + c &= -2 \\ 25a + 5b + c &= 64 \end{aligned}$$

Solving:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & 1 & -2 \\ 25 & 5 & 1 & 64 \end{array}\right] \xrightarrow[-25R_1+R_3]{-1 \cdot R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -6 \\ 0 & -20 & -24 & -36 \end{array}\right]$$

$$\xrightarrow{-10R_2+R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & 0 & -6 \\ 0 & 0 & -24 & 24 \end{array}\right]$$

$-24c = 24$ $\quad$ $-2b = -6$ $\quad$ $a + b + c = 4$

$c = -1$ $\quad$ $b = 3$ $\quad$ $a + 3 - 1 = 4$

$a + 2 = 4$

$a = 2$

The function is

$$\boxed{y = 2x^2 + 3x - 1}$$

(3) $A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix}$

The row operation $1 \cdot R_1 + R_2$ is achieved by left multiplication by $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The row operation $-4 \cdot R_1 + R_3$ is achieved by left multiplication by $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

Doing the first row operation and then the second is achieved by left multiplication by $E = E_{31} E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.

Check:

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 8 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

(4) A vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that is perpendicular to the vectors $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$ will have

both $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 0$ and $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = 0$.

Rewriting, we want to solve the system

$$x + 2y + 4z = 0$$
$$5x + 3y + z = 0.$$

Solving:

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 5 & 3 & 1 & 0 \end{array}\right] \xrightarrow{-5\cdot R_1 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -7 & -19 & 0 \end{array}\right]$$

We may generate solutions by choosing the value for z arbitrarily:

$$-7y - 19z = 0$$
$$\boxed{y = \frac{19}{-7} z}$$

$$x + 2y + 4z = 0$$
$$x + \frac{-38}{7} z + 4z = 0$$
$$x - \frac{10}{7} z = 0$$
$$\boxed{x = \frac{10}{7} z}$$

(*or, all multiples of $(10, -19, 7)$.)

(5) $A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{bmatrix}$

We find A^{-1} by Gauss - Jordan Elimination.

$$[\, A \mid I \,]$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 4 & 5 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-2\cdot R_1 + R_3]{-1R_1 + R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 3 & -5 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -5 & -2 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[\frac{3}{2}R_3 + R_1]{-\frac{5}{2}R_3 + R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & -\frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 3 & 0 & \frac{1}{2} & -\frac{5}{2} & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{3}R_2+R_1} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & -\frac{2}{3} & \frac{7}{3} & -\frac{1}{3} \\ 0 & 3 & 0 & \frac{1}{2} & -\frac{5}{2} & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{2}R_3]{\frac{1}{2}R_1,\ \frac{1}{3}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{7}{6} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{6} & -\frac{5}{6} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{array}\right]$$

So

$$A^{-1} = \frac{1}{6}\begin{bmatrix} -2 & 7 & -1 \\ 1 & -5 & 2 \\ 3 & -3 & 0 \end{bmatrix}$$

Ex (a) $A\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\vec{x} = A^{-1}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{6} \\ \frac{1}{2} \end{bmatrix}$$

etc ...