Homework Assignment 3

Due Wednesday 9/24/2014 at start of class.

Assigned:
Section 2.7: 1, 2, 4, 7, 8, 9, 10, 11, 13, 19, 38, 40
Section 3.1: 1, 2, 4, 5, 10, 12, 14, 15, 18, 19, 20, 23, 25, 26–32

Collected:

(1) (a) What are the three elementary row operations?

(b) When doing elimination, why are you permitted to do the three elementary row operations? Explain for each operation. Why don't we do "column operations"?

(2) Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

Find a lower triangular matrix L and an upper triangular matrix U so that

$$A = LU.$$

(3) (a) Write down all 3×3 permutation matrices.

(b) Which permutation matrix P does this:

$$P\vec{i} = \vec{j}, \qquad P\vec{j} = \vec{k}, \qquad P\vec{k} = \vec{i} \qquad ?$$

(c) The matrix P naturally defines a transformation T of $\mathbb{R}^3$ to itself:

$$T\colon \mathbb{R}^3 \to \mathbb{R}^3, \qquad T(\vec{x}) = P\vec{x}.$$

Geometrically, what is the transformation T doing to all vectors in $\mathbb{R}^3$?

(d) Describe the transformation defined by the matrix P^T.

(4) Decide which of the following sets are subspaces of the given vector space. As always, *convince the reader*.

(a) Is $\{(x, y) \,|\, x, y \geq 0 \text{ or } x, y \leq 0\}$ a subspace of $\mathbb{R}^2$?

(b) Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^n$. Is the set of all vectors perpendicular to these two a subspace of $\mathbb{R}^n$?

(c) Is $\{(a_1, a_2, \ldots, a_n) \,|\, \sum_{i=1}^{n} a_i = 0\}$ a subspace of $\mathbb{R}^n$?

(d) Is $\{(a_1, a_2, \ldots, a_n) \,|\, \sum_{i=1}^{n} a_i = 1\}$ a subspace of $\mathbb{R}^n$?

(e) Is the set of all symmetric 2×2 matrices a subspace of the space of all 2×2 matrices?

(f) Take a finite set of vectors in $\mathbb{R}^n$; say, $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}$. Is the set

$$S = \{c_1\vec{v_1} + c_2\vec{v_2} \cdots + c_k\vec{v_k} \,|\, c_i \in \mathbb{R}\}$$

a subspace of $\mathbb{R}^n$? (S is the set of *all linear combinations* of the given vectors.)

(5) (a) Explain the famous *Fibonacci sequence*: $1, 1, 2, 3, 5, 8, 13, 21, \cdots$.

(b) Let a_n denote the nth Fibonacci number ($n \geq 1$). Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Carefully explain why

$$A \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} a_{n+2} \\ a_{n+1} \end{bmatrix}.$$

(c) Find A^{-1} and compute $A^{-1}\begin{bmatrix} 13 \\ 8 \end{bmatrix}$. Does this make sense?

As you know, the matrix A can be understood as defining a transformation of $\mathbb{R}^2$ to itself. Later this semester we will use the geometry of this transformation to unlock the mysteries of the Fibonacci numbers. Stay tuned.

Bonus! For each problem that you successfully complete below, I will add one point to this homework's score. The standard by which I will grade these is higher than usual. Your solution must be completely clear and error-free or you will not receive any credit.

(1) Consider the following set of matrices:

$$\mathcal{C} = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

(a) Show that $\mathcal{C}$ is a subspace of the space of all 2×2 matrices.

(b) Suppose that $A \in \mathcal{C}$ and $B \in \mathcal{C}$. Show that even $AB \in \mathcal{C}$. (This is unusual.)

(c) This particular set of matrices forms an algebraic structure that you have encountered before. What structure am I talking about?

Hint: The matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *is in* $\mathcal{C}$*. What happens when you square this matrix?*

(2) This exercise will show me how clever you are constructing 2×2 matrices, and how well you understand matrix multiplication. Start with the matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Your job is to find 2×2 matrices B and C so that BAC is a diagonal matrix. One more restriction: I want B and C to have integer entries only (no fractions). Have some confidence and try it!

Hint: This problem is easy without the integer restriction. In that case, you would only need the matrix B since the diagonal matrix could be achieved through row operations alone. Try this to see what I mean, and notice the fractions that appear. To get around the fractions, do some row operations with a matrix B and then do some "column operations" with a matrix C.