Name:
Pledge:

Math 435 Final Exam

You may use any and all of your resources for this exam. I only ask that you do not discuss it with each other. In other words, the only human who should give you any assistance on this exam is Prof. Ducey. Please slide it under my office door, Roop 339, by Friday 12/18/2015, 12:30pm.

Do **all** of the problems below. (10 points each)

(1) Let A_0 be the closed interval $[0,1]$ in $\mathbb{R}$. For $n > 0$, define

$$A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left(\frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right).$$

So A_1 is obtained from A_0 by deleting the middle third $(\frac{1}{3}, \frac{2}{3})$. Then A_2 is obtained from A_1 by deleting the two "middle thirds" $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$. And so on. Finally, set

$$\mathcal{C} = \bigcap_{n=0}^{\infty} A_n.$$

The set $\mathcal{C}$ is called the *Cantor Set*, and is a famous example/counterexample in topology, analysis, and measure theory.

(a) Show that $\mathcal{C}$ is totally disconnected (i.e., the only connected subspaces are the one-point sets).

(b) Show that $\mathcal{C}$ is compact.

(c) Show that each set A_n is a union of finitely many disjoint closed intervals of length $1/3^n$, and show that the end points of these intervals lie in $\mathcal{C}$.

(d) Show that $\mathcal{C}$ has no isolated points (i.e., one-point sets that are open).

(e) Conclude that $\mathcal{C}$ is uncountable. (You may just quote Theorem 27.7.)

(2) Let X be metrizable. Show that X is second countable (and so Lindelöf and separable) if and only if every discrete subspace of X is countable.

(3) Let (X, d) be a separable metric space. Let $\{x_n\}$ be a countable dense subset. Consider the map $f\colon X \to \mathbb{R}^\omega$, where the range is given the product topology, defined by

$$f(x) = (d(x, x_1), d(x, x_2), \cdots).$$

(a) Show that f is injective and continuous.

(b) Is f an imbedding?

Bonus! (2 points each)

- (2 points) Show that a closed subspace of a normal space is normal.
- (2 points) Find an example of a compact subspace of a space X whose closure in X is not compact.
- Let X be a space. A subspace A is called a *retract* of X if there exists a continuous map $r\colon X \to A$ with the property that $r(a) = a$ for all $a \in A$.
  - (2 points) Show that S^1 is a retract of the punctured plane $\mathbb{R}^2 \backslash \{0\}$.
  - (2 points) Suppose that X has the property that any continuous map $X \to X$ has a fixed point. Show that if A is a retract of X, then A also has this property.