Math 435
Homework Assignment 3

Due Wednesday 10/7/2015

(1) Prove that a map $f\colon X \to Y$ of topological spaces is continuous iff $f^{-1}(F)$ is a closed subset of X for each closed subset F of Y.

(2) (2 pts) Prove that the standard topology on $\mathbb{R}^n$ has a countable basis.

(3) Let X be a topological space, let $A \subset X$. Suppose that, for every point $a \in A$ there exists a neighborhood U of a that is contained in A. Prove that A is open.

(4) Following the definition of open sets in metric spaces, prove that the collection of open sets in a metric space (X, d) gives a topology on X.

(5) (2 pts) Let $\{\tau_i\}$ be a family of topologies on a set X. Prove that $\cap_i \tau_i$ is a topology on X. What about $\cup_i \tau_i$?

(6) (3 pts) Let X be a set. Let

$$\tau_f = \{U \subset X \mid X \backslash U \text{ is finite or all of } X\};$$
$$\tau_c = \{U \subset X \mid X \backslash U \text{ is countable or all of } X\};$$
$$\tau_\infty = \{U \subset X \mid X \backslash U \text{ is infinite or empty or all of } X\}.$$

Are $\tau_f, \tau_c, \tau_\infty$ always topologies on X?

Bonus Problems

(1) Let X, Y, S be sets. Let $\phi(X) = \{\text{functions from } X \text{ to } S\}$, $\phi(Y) = \{\text{functions from } Y \text{ to } S\}$. Given a map $f\colon X \to Y$, define the map

$$f^*\colon \phi(Y) \to \phi(X), \quad f^*(h) = hf.$$

Suppose that you are given a left inverse $g\colon Y \to X$, so that $gf = i_X$. Show that $f^* g^* = i_{\phi(X)}$. (This exercise gives a useful trick for when you can say something about left inverses, but want to say something about right inverses.)

(2) (For those who know some algebra.) This example is very important in algebraic geometry. Let R be a commutative ring, and let $\mathrm{Spec}R$ be the set of all prime ideals of R. Given an ideal I of R, let $F_I \subset \mathrm{Spec}R$ be the set of all prime ideals containing I. Prove that the family

$$\{\mathrm{Spec}R \backslash F_I \mid I \text{ is an ideal of } R\}$$

is a topology on $\mathrm{Spec}R$.