Some open problems in algebraic combinatorics.

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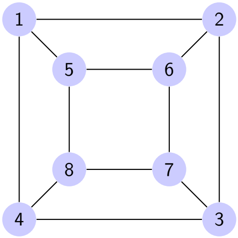
Math & Stat Colloquium
James Madison University

October 25, 2022

Outline

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$



- Γ, a finite simple graph with adjacency matrix A.

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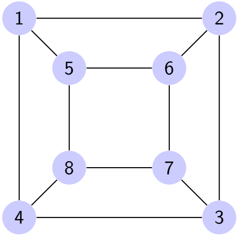

- Γ, a finite simple graph with adjacency matrix A.
- The matrix encodes the graph, we can study the matrix using linear algebra.

Strongly regular graphs

A *strongly regular graph* (SRG) with parameters v, k, λ, μ:

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- It follows:

$$A^2 = kI + \lambda A + \mu(J - A - I)$$

Diamonds in the rough

Prev Up Next

	v	k	λ	μ	r^f	s^g	comments
!	153	32	16	4	14^{17}	-2^{135}	Triangular graph T(18)
		120	91	105	1^{135}	-15^{17}	pg(8,14,7)
?	153	56	19	21	5^{84}	-7^{68}	pg(8,6,3)?
		96	60	60	6^{68}	-6^{84}	
?	153	76	37	38	5.685^{76}	-6.685^{76}	2-graph*?
?	154	48	12	16	4^{98}	-8^{55}	pg(6,7,2)?
		105	72	70	7^{55}	-5^{98}	
-	154	51	8	21	2^{132}	-15^{21}	Krein2
		102	71	60	14^{21}	-3^{132}	Krein1
?	154	72	26	40	2^{132}	-16^{21}	
		81	48	36	15^{21}	-3^{132}	
+	155	42	17	9	11^{30}	-3^{124}	S(2,3,31); lines in PG(4,2)
		112	78	88	2^{124}	-12^{30}	
+	156	30	4	6	4^{90}	-6^{65}	O(5,5) Sp(4,5); GQ(5,5)
		125	100	100	5^{65}	-5^{90}	
+	157	78	38	39	5.765^{78}	-6.765^{78}	Paley(157); 2-graph*
?	160	54	18	18	6^{75}	-6^{84}	pg(9,5,3) does not exist (no 2-graph* for line graph)
		105	68	70	5^{84}	-7^{75}	
-	161	80	39	40	5.844^{80}	-6.844^{80}	Conf
?	162	21	0	3	3^{105}	-6^{56}	

Rearranging, we have

$$A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J.$$

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The all-one vector is an eigenvector for A with eigenvalue k. An eigenvector $\vec{x}$ with a different eigenvalue $c \neq k$ will satisfy:

$$\vec{0} = (A^2 + (\mu - \lambda)A + (\mu - k)I)\vec{x} = (c^2 + (\mu - \lambda)c + (\mu - k))\vec{x},$$

and so

$$c^2 + (\mu - \lambda)c + (\mu - k) = 0.$$

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Denote the roots by r, s, with respective multiplicities as eigenvalues f, g. So the eigenvalues of A are:

$$r^f, \quad s^g, \quad k.$$

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Finally, we have

$$\begin{aligned} \mathrm{tr}(A) = 0 &= fr + gs + k \\ v &= f + g + 1. \end{aligned}$$

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$$v \leq 1 + k + k(k-1) + k(k-1)^2 + \cdots + k(k-1)^{d-1}.$$

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A Moore graph with degree k and diameter 2 satisfies:

$$v = 1 + k + k(k-1) = k^2 + 1.$$

Such a graph is an $SRG(v, k, 0, 1)$.

Some calculations

In this case $c^2 + c + 1 - k = 0$ has solutions

$$r = \frac{-1 + \sqrt{4k-3}}{2}, \quad s = \frac{-1 - \sqrt{4k-3}}{2}.$$

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From $0 = fr + gs + k$ we get

$$0 = f\left(\frac{-1 + \sqrt{4k-3}}{2}\right) + g\left(\frac{-1 - \sqrt{4k-3}}{2}\right) + k$$

$$0 = (f+g) - (f-g)\sqrt{4k-3} - 2k$$

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$$\begin{aligned}
0 &= (f+g) - (f-g)\sqrt{4k-3} - 2k \\
0 &= k^2 - (k^2 - 2g)t - 2k \\
0 &= \frac{1}{16}(t^2+3)^2 - \left[\frac{1}{16}(t^2+3)^2 - 2g\right] t - \frac{1}{2}(t^2+3) \\
&\vdots
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So in this case t divides 15.

Hoffman-Singleton Theorem, 1960

Theorem

Let Γ be a Moore graph with degree k and diameter 2. Then

$$k \in \{2, 3, 7, 57\}.$$

$k = 2$: pentagon

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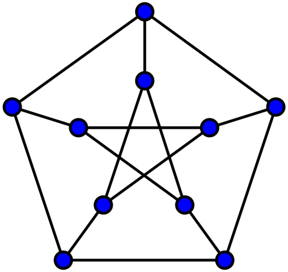
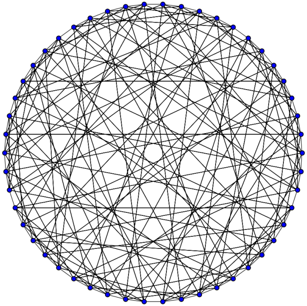
$k = 3$: Petersen graph



$k = 7$: Hoffman-Singleton graph (image from Wikipedia)



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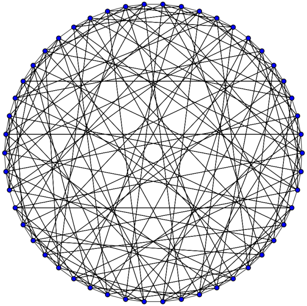
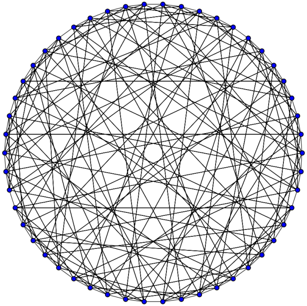


$k = 57$:

$k = 7$: Hoffman-Singleton graph (image from Wikipedia)



$k = 57$: None ever found

Student bragging

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Math 485 (2021), determined the Smith group

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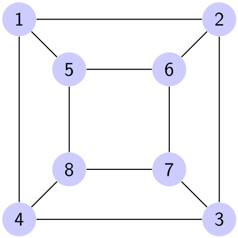
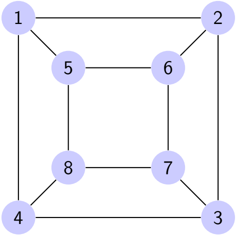


$$L = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & -1 & 3 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 3 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$



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$$M\colon \mathbb{Z}^n \to \mathbb{Z}^m$$

The cokernel of this map $\mathbb{Z}^m/\operatorname{Im}(M)$ is a finitely generated abelian group.

Examples: integer invariants

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Most of the invariants of other subset incidence relations remain unknown.

Student bragging

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All such invariants for 3-subsets are unknown.

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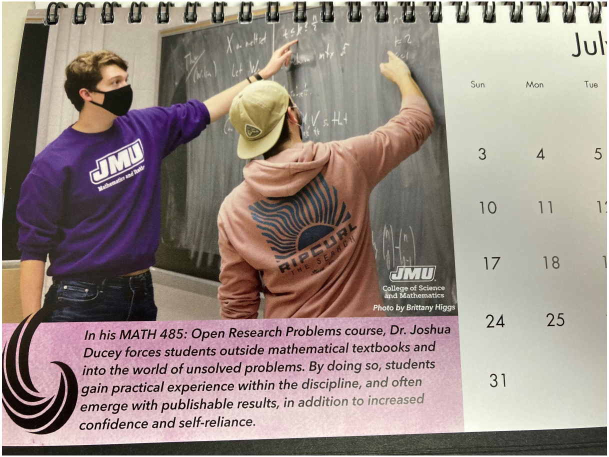

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The n-cube graph Q_n

Vertices:

$$\{(a_1, a_2, \cdots, a_n) \mid a_i = 0 \text{ or } 1\}$$

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Edges: two vertices are adjacent if they differ in exactly one position.

Work of Bai, Jacobson-Niedermeier-Reiner, and others show that the Laplacian integer invariants (i.e., sandpile group) can be understood by the p-primary components, for all primes except $p = 2$.

Student bragging

Deelan Jalil (2013) recovered these results in a novel way. Also found the invariants of the adjacency matrix in many cases, and conjectured the Smith group structure in general. Conjecture was later proved by Chandler-Sin-Xiang (2017).

Sandpile group of Q_n: $\kappa(Q_n)$

For $p \neq 2$,

$$Syl_p(\kappa(Q_n)) \cong Syl_p\left(\oplus_{j=1}^{n} (\mathbb{Z}/2j\mathbb{Z})^{\binom{n}{j}}\right)$$

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Still not even a conjecture for $Syl_2(\kappa(Q_n))$.

Thank you for your attention!