The Smith Normal Form of the Incidence Matrix of Skew Lines in $\mathrm{PG}(3, q)$

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In memory of Bob Liebler

Outline

Smith normal form

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- More precisely, if A is an $m \times n$ integer matrix, then there exist unimodular (invertible over the integers) matrices P and Q such that the matrix $PAQ = (d_{i,j})$ satisfies

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- The diagonal entries of the Smith normal form of A are unique up to sign, and are called the invariant factors of the matrix A. The largest prime powers dividing the invariant factors are called the elementary divisors of A.

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- $\mathcal{L}_r$ vs. $\mathcal{L}_s$; various possible notions of incidence.
- Consider the incidence relation of *zero-intersection*, and let $A_{r,s}$ denote the $\left[{n \atop r}\right]_q \times \left[{n \atop s}\right]_q$ incidence matrix.

Some History (zero-intersection)

- (1968) Hamada gives formula for p-rank of $A_{r,s}$ when $r = 1$.
- (1980) Lander gives SNF of points vs. lines in $PG(2, q)$.
- (1990) Black and List compute SNF of points vs. hyperplanes when $q = p$.
- (2000) Sin gives SNF of points vs. s-subspaces when $q = p$.
- (2002) Liebler and Sin work out SNF of points vs. hyperplanes for arbitrary q.
- (2004) Sin computes p-ranks for r-subspaces vs. s-subspaces.
- (2006) Chandler, Sin, Xiang give SNF of points vs. s-subspaces for arbitrary q.
- Natural to consider when V is 4-dimensional over $\mathbb{F}_q$, incidence of $\mathcal{L}_2$ vs. $\mathcal{L}_2$; i.e. skew lines in $\mathrm{PG}(3, q)$.

SRG equation

- $A = A_{2,2}$ satisfies

$$A^2 = q^4 I + (q^4 - q^3 - q^2 + q)A + (q^4 - q^3)(J - A - I)$$

- A has eigenvalues q, $-q^2$, q^4 with respective multiplicities $q^4 + q^2, \quad q^3 + q^2 + q, \quad$ and 1
- $|\det(A)| = q^{q^4+2q^3+3q^2+2q+4}$
- In particular, all the invariant factors of A are powers of p

Example: skew lines in $\mathrm{PG}(3,4)$

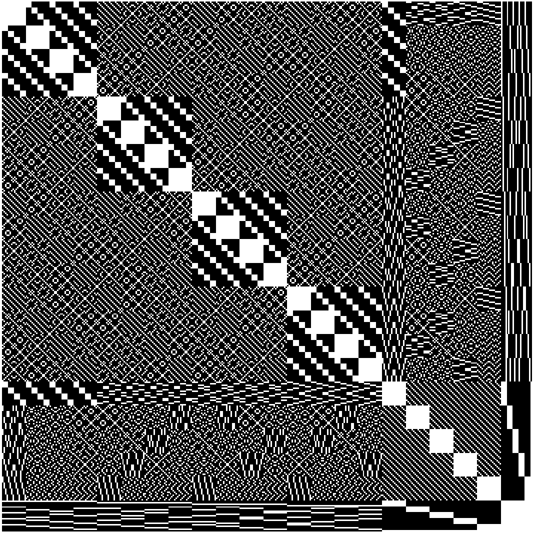

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Table: The elementary divisors of the incidence matrix of lines vs. lines in $\mathrm{PG}(3,4)$, where two lines are incident when skew.

Elem. Div.	1	2	2^2	2^3	2^4	2^5	2^6	2^7	2^8
Multiplicity	36	16	220	0	32	16	36	0	1

Table: LinBox computations for some small values of $q = p^t$.

	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	e_9	e_{10}	e_{11}	e_{12}	$\cdots$
$q = 2$	6	14	8	6	1									
$q = 3$	19	71	20	19	1									
$q = 5$	85	565	70	85	1									
$q = 7$	231	2219	168	231	1									
$q = 2^2$	36	16	220		32	16	36		1					
$q = 3^2$	361	256	6025		202	256	361		1					
$q = 2^3$	216	144	96	3704			128	96	144	216			1	

Here e_i denotes the multiplicity of p^i as an elementary divisor of A. An empty entry in the table denotes a 0.

Theorem (Brouwer–D–Sin, 2011)

Let e_i denote the number of times p^i occurs in the Smith normal form of A. Then, for $0 \leq i \leq t$,

$$e_{2t+i} = \sum_{\vec{s} \in \mathcal{H}(i)} d(\vec{s}).$$

Notation key:

- $\mathcal{H}(i) = \left\{(s_0, \ldots, s_{t-1}) \in [3]^t \;\middle|\; \#\{j | s_j = 2\} = i\right\}$.

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- $d(\vec{s}) = \prod_{i=0}^{t-1} d_{\lambda_i}$.

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1. Smith normal form
2. Incidence of Subspaces
3. Incidence maps and SNF bases

- R is a discrete valuation ring, maximal ideal generated by p.
- $\eta\colon R^m \to R^n$
- $M_i = \{x \in R^m \,|\, \eta(x) \in p^i R^n\}$
- $N_i = \{p^{-i}\eta(x) \,|\, x \in M_i\}$
- Set $F = R/pR$. $\overline{L} = (L + pR^\ell)/pR^\ell$ is an F-vector space.
- $e_i = \dim_F \left(\overline{M_i}/\overline{M_{i+1}}\right) = \dim_F \left(\overline{N_i}/\overline{N_{i-1}}\right)$

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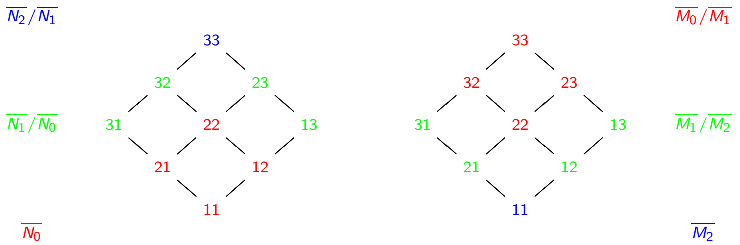
- $A_{2,1}A_{1,2} = -[A+(q^2-q)I] + q^2 I + (q^3+q^2-q)J$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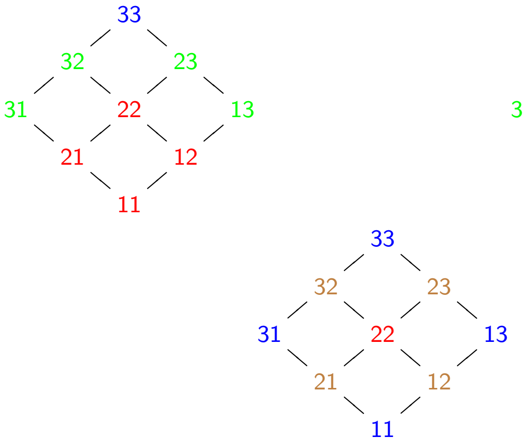
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Thank you for your attention!