Exam 1 Part one.

True or False?

1. Every cyclic group is abelian.

   This statement is true. If $G = \langle a \rangle$ is cyclic, then all elements of G are of the form a^k, for some $k \in \mathbb{Z}$. We have

   $$a^i a^j = a^{i+j} = a^j a^i,$$

   so G is abelian.

2. Every abelian group is cyclic.

   This statement is false. Some counterexamples are $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{C}$ (all under addition), $U(8)$, the subgroup $\{R_0, R_{180}, H, V\}$ of D_4, etc.

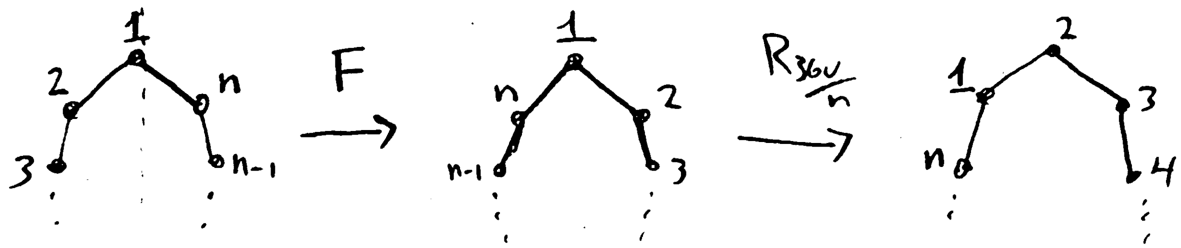
3. If G is an infinite group and H is a nontrivial subgroup of G, then H is infinite.

This statement is <u>false</u>. Some counterexamples are $\{-1, 1\} \leq \mathbb{R}^*$, $\{-1, 1, i, -i\} \leq \mathbb{C}^*$, or more generally the subgroup of n^{th} roots of unity in $\mathbb{C}^*$.

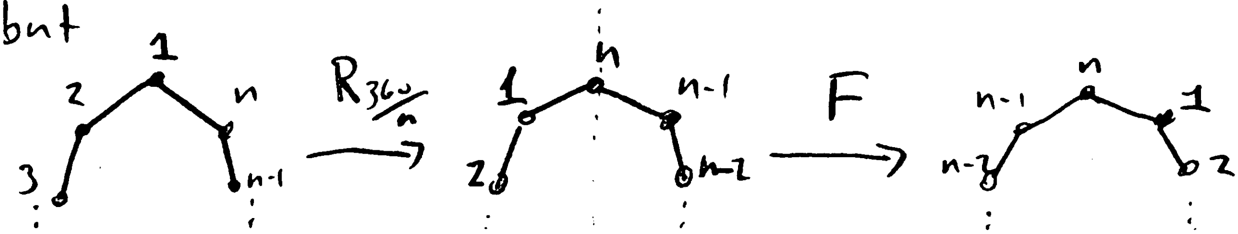
4. For all $n \geq 3$, the dihedral group D_n is nonabelian.

This statement is <u>true</u>. Let F denote a reflection across the line of symmetry through a vertex, and as usual let $R_{360/n}$ denote counterclockwise rotation by $\frac{360}{n}$ degrees.

Then $F R_{360/n} \neq R_{360/n} F$, since



but



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Part 2.

1. G is a group, $H \leq G$, and $g \in G$. We show that $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$ is a subgroup of G.

Pf Since H is nonempty so is gHg^{-1}.

Let $a, b \in gHg^{-1}$. Then $a = gh_1g^{-1}$ and $b = gh_2g^{-1}$, for some $h_1, h_2 \in H$. We have

$$ab = (gh_1g^{-1})(gh_2g^{-1}) = gh_1h_2g^{-1}$$

and this is an element of gHg^{-1} since $h_1h_2 \in H$.

This shows gHg^{-1} is closed under the operation.

Finally we check that if $c \in gHg^{-1}$, then $c^{-1} \in gHg^{-1}$.

If $c \in gHg^{-1}$ then $c = ghg^{-1}$ for some $h \in H$.

Then $c^{-1} = (ghg^{-1})^{-1} = gh^{-1}g^{-1}$ and this is an element of gHg^{-1} since $h^{-1} \in H$.

By the two-step subgroup test, gHg^{-1} is a subgroup of G.

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2. G is a group, $H \leq G$, and $g \in G$. Set $gH = \{gh \mid h \in H\}$. We show that if $g \notin H$ then $gH \cap H = \emptyset$.

PB Probably easier to prove the contrapositive of this statement. So we will assume that $gH \cap H \neq \emptyset$ and show that $g \in H$.

Suppose $x \in gH \cap H$. Since $x \in gH$, we can write

$x = gh$, for some $h \in H$.

Then $g = xh^{-1}$ and this is an element of H since both x and h^{-1} are in H.

<u>Bonus</u>

See the back of the textbook for a Cayley table of D_3.