Name:
Pledge:

Math 434 Final Exam

You may use all of your resources for this final exam. However, the only *human* that should help you with this exam is me.

Do all of the problems below. (10 points each.)

1. Let V be a finite dimensional inner product space. Suppose $T \in \mathcal{L}(V)$ has the following property: $\langle v, u \rangle = \langle Tv, Tu \rangle$ for all $v, u \in V$.

   - Prove that T is invertible and describe its inverse.

2. Consider a set of four points $\{p_1, p_2, p_3, p_4\}$. Our vector space in this problem is the space of all functions from the set of points to the field:

   $$V = \{f \colon \{p_1, p_2, p_3, p_4\} \to \mathbb{F}\}.$$

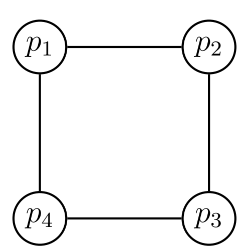
   For $1 \leq i \leq 4$, define the element $e_i \colon \{p_1, p_2, p_3, p_4\} \to \mathbb{F}$ of V by the formula:

   $$e_i(p_k) = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

   So, for example, $e_1(p_1) = 1$ but $e_1(p_2) = e_1(p_3) = e_1(p_4) = 0$.

   - Show that e_1, e_2, e_3, e_4 is a basis of V.

3. Consider again the four points $\{p_1, p_2, p_3, p_4\}$ but this time arranged in a square. For definiteness, say there is an edge connecting the following pairs of points: $(p_1, p_2), (p_2, p_3), (p_3, p_4), (p_4, p_1)$.



We will use the edges to define a linear operator on V. Define $T\colon V \to V$, $f \mapsto Tf$, by the following formula:

$$(Tf)(x) = \sum_y f(y),$$

where the sum is taken over all points y that are connected by an edge to x. As an example, $(Tf)(p_1) = f(p_2) + f(p_4)$. In this way, any function $f\colon \{p_1, p_2, p_3, p_4\} \to \mathbb{F}$ determines a function $Tf\colon \{p_1, p_2, p_3, p_4\} \to \mathbb{F}$.

- Show that T is indeed a linear transformation from V to V.
- Determine the matrix of T with respect to the basis e_1, e_2, e_3, e_4 of the previous problem.

Bonus! (1 point each)

- Same notation as in problems (2) and (3) above. Show that the element $f \in V$ defined by

  $$f(p_1) = f(p_2) = f(p_3) = f(p_4) = 1$$

  is an eigenvector for the operator T. What property of the square is reflected in the corresponding eigenvalue?

- What is the dimension of range(T)?

- Tell me something you learned from one of the presentations. (Not your own presentation.)