Invariants of an Incidence Matrix Related to Rota's Basis Conjecture

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July 20, 2012

Rota's Basis Conjecture (again)

- Recall Rota's Basis Conjecture states that given a set of n bases for an $n-\textit{dimensional}$ vector space, one can always make n, disjoint bases each containing one vector from each of the original bases.

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- Recall Rota's Basis Conjecture states that given a set of n bases for an $n - dimensional$ vector space, one can always make n, disjoint bases each containing one vector from each of the original bases.
- We can think of these in terms of "rainbow" sets of vectors.

$$\begin{matrix} {\color{red}a_1} & {\color{red}a_2} & {\color{red}\cdots} & {\color{red}a_n} \\ {\color{blue}b_1} & {\color{blue}b_2} & {\color{blue}\cdots} & {\color{blue}b_n} \\ \vdots & \vdots & & \vdots \\ {\color{green}k_1} & {\color{green}k_2} & {\color{green}\cdots} & {\color{green}k_n} \end{matrix}$$

- We will use *transversal* to refer to any rainbow set of vectors whether they form a basis or not.

The Incidence Matrix for Disjointness of Transversals

- Consider the matrix A_n which has rows and columns indexed by transversals (rainbow sets of vectors) of dimension n, and has a 1 in the i, jth spot if the transversal at i and the transversal at j are disjoint and a 0 otherwise.

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- A_n may also be called the adjacency matrix for the graph in which the transversals are the vertices, connected by an edge if they are disjoint.
- A_n has dimension $n^n \times n^n$

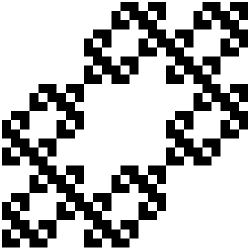
Here is an example for $n = 2$

$$\begin{array}{c} \\ (a_1, b_1) \\ (a_1, b_2) \\ (a_2, b_1) \\ (a_2, b_2) \end{array} \begin{array}{c} \begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & (a_2, b_1) & (a_2, b_2) \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

A_n for $n = 3$ and $n = 5$

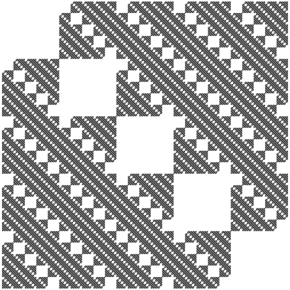
$n = 3$

(27×27)



$n = 5$

(3125×3125)



Smith Normal Form

- The Smith Normal Form (SNF) is a diagonal form of an integer matrix that can be obtained by multiplying the matrix on the left and right sides by unimodular ($determinant = \pm 1$) invertible square integer matrices.

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- The Smith Normal Form (SNF) is a diagonal form of an integer matrix that can be obtained by multiplying the matrix on the left and right sides by unimodular ($\mathit{determinant} = \pm 1$) invertible square integer matrices.
- The SNF can be defined for any matrix with entries in the integers, such as A_n
- The entries on the diagonal of SNF are called the *invariant factors*. They will also be integers, with each entry being a divisor of the subsequent ones.

Smith Normal Form continued

Over a field

$$PXQ = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Over the integers

$$PXQ = \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_N \end{pmatrix}$$

Smith Normal Form continued

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Here is a non-square example

$$X = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix}, Q = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$SNF(X) = PXQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

Why Smith Normal Form

- The SNF is unique for a given matrix and is not affected by permuting the columns/rows

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- The SNF is unique for a given matrix and is not affected by permuting the columns/rows
- Determinants of submatrices. d_k indicates the GCF of all submatrices of dimension $k \times k$ of the matrix.

$$X = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

$$s_1 = d_1$$

$$s_1 s_2 = d_2$$

$$\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3, \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = -6, \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3$$

$$\vdots$$

$$s_1 s_2 \dots s_N = d_N$$

$$SNF(X) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

Dimensions 2, 3, 4, and 5

- We computed the SNF of A_n for $n = 2, 3, 4, 5$
- The first entry in the ordered pair denotes the value of the integer that appears in SNF and the second entry denotes its multiplicity (i.e. $(1, 10)$ indicates that 1 appears on the SNF diagonal 10 times.)

$$\text{So } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \text{ would be } ((1, 1), (3, 1))$$

Dimension	SNF
2	((1,4))
3	((1,8),(2,12),(4,6),(8,1))
4	((1,81),(3,108),(9,54),(27,12),(81,1))
5	((1,1024),(4,1280),(16,640),(64,160),(256,20),(1024,1))

A Pattern

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- Now we multiply the entries of the ordered pairs together and factor.

Dimension	
2	$(1^{2}\binom{2}{0}, 1^{2}\binom{2}{1}, 1^{2}\binom{2}{2})$
3	$(2^{3}\binom{3}{0}, 2^{3}\binom{3}{1}, 2^{3}\binom{3}{2}, 2^{3}\binom{3}{3})$
4	$(3^{4}\binom{4}{0}, 3^{4}\binom{4}{1}, 3^{4}\binom{4}{2}, 3^{4}\binom{4}{3}, 3^{4}\binom{4}{4})$
5	$(4^{5}\binom{5}{0}, 4^{5}\binom{5}{1}, 4^{5}\binom{5}{2}, 4^{5}\binom{5}{3}, 4^{5}\binom{5}{4}, 4^{5}\binom{5}{5})$

Conjecture About the SNF of A_n

The Smith Normal Form of A_n which denotes the incidence matrix for disjointness of transversals for dimension n will have entries given by

$$((n-1)^k, (n-1)^{n-k}\binom{n}{k})$$

where k indicates we are referring to the kth ordered pair (starting at $k=0$ and ending at $k=n$)

The Eigenvalues of A_n

- Graph theorists interested in eigenvalues of incidence matrices
- The eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_k$ of any matrix are related to the invariant factors $s_1, s_2, \ldots, s_k$ by the fact that $\prod_{i=1}^{k} \lambda_i = \prod_{i=1}^{k} s_i$
- Data suggest the eigenvalues of A_n are the conjectured invariant factors up to sign, i.e. we suspect that the eigenvalues of A_n all equal $\pm(n-1)^k$ for $k = 0, 1, \ldots, n$

Recall...

- We conjectured that the invariant factors of A_n are given by $(n-1)^k$ each with multiplicity $(n-1)^{n-k}\binom{n}{k}$.

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Observe...

- For a fixed transversal $z = (a_i, b_j, \ldots, c_l)$, $(n-1)^{n-k}\binom{n}{k}$ is the number of transversals with exactly k elements in common with z.
- What happens when we use this fact to index the rows and columns of A_n?

Then A_n looks like this...

$$A_n = \begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,n} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,0} & A_{n,1} & \cdots & A_{n,n} \end{pmatrix}$$

- Block $A_{i,j}$ contains rows indexed by transversals with i elements in common with some fixed z and columns indexed by transversals with j elements in common with z
- $A_{i,j}$'s have constant row sums $(n-1)^i(n-2)^{(n-i-j)}\binom{n-i}{j}$

Lemma

(Van Lint)

Let M be a matrix of size m by m which has the form

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,k} \\ \vdots & \ddots & \ldots & \vdots \\ M_{k,1} & M_{k,2} & \cdots & M_{k,k} \end{pmatrix}$$

Where each $M_{i,j}$ is a submatrix of size m_i by $m_j (i = 1, 2, \ldots, k; j = 1, 2, \ldots, k)$. Suppose that for each i and j the matrix $M_{i,j}$ has constant row sums b_{ij}. Let B be the matrix with entries b_{ij}. Then each eigenvalue of B is an eigenvalue of M.

A_3 indexed with respect to disjointness

$$
\left(\begin{array}{cccccccc|cccccccccccc|ccccc}
0&0&0&0&0&0&0&1&1&0&0&0&1&0&1&0&0&0&0&0&1&0&1&0&1\\
0&0&0&0&0&0&1&0&0&1&0&0&0&1&1&0&0&0&0&0&0&1&1&0&1\\
0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&1&0&0&0&0&1&0&0&1&1\\
0&0&0&0&1&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0&1&1\\
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1&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0\\
\hline
1&0&0&0&1&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&1\\
0&1&0&0&0&1&0&0&0&0&0&0&1&0&0&1&1&0&0&1&0&0&0&0&1\\
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0&0&0&0&1&1&0&0&0&0&1&1&1&1&0&0&0&0&0&0&1&1&0&0&0\\
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1&1&0&0&1&1&0&0&0&0&0&0&1&1&0&0&1&1&0&0&0&0&0&0&0\\
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0&0&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
\hline
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{array}\right.
$$

Matrix of row sums

$$B_3 = \begin{pmatrix} 1 & 3 & 3 & 1 \\ 2 & 4 & 2 & 0 \\ 4 & 4 & 0 & 0 \\ 8 & 0 & 0 & 0 \end{pmatrix}$$

- We can make this upper triangular by permuting the columns

Theorem

For each $k \in \{0, 1, \ldots, n\}$, $(n-1)^k$ is an eigenvalue of A'_n where A'_n is a matrix obtained by reversing the column order of A_n

Proof

Recall we indexed A_n so that

$$A_n = \begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,n} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,0} & A_{n,1} & \cdots & A_{n,n} \end{pmatrix}$$

Where the i,j-th block has constant row sum $(n-1)^i (n-2)^{(n-i-j)} \binom{n-i}{j}$

Proof

Continued...

Thus, upon reversing column order we have

$$A'_n = \begin{pmatrix} A_{0,n} & A_{0,n-1} & \cdots & A_{0,0} \\ A_{1,n} & A_{1,n-1} & \cdots & A_{1,0} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,n} & A_{n,n-1} & \cdots & A_{n,0} \end{pmatrix}$$

Where the i, j-th block has constant row sum $(n-1)^i(n-2)^{(j-i)}\binom{n-i}{n-j}$

Proof

So if $i > j$ *then* $n-j > n-i$ *so* $\binom{n-i}{n-j} = 0$ *so* $b_{ij} = (n-1)^i(n-2)^{(j-i)}\binom{n-i}{n-j} = 0$. *And if* $i = j$ *Then* $b_{ij} = (n-1)^i(n-2)^{(j-i)}\binom{n-i}{n-j} = (n-1)^i(n-2)^0\binom{n-i}{n-i} = (n-1)^i$. *Thus* B_n *is upper-triangular with diagonal entries* $(n-1)^i$, *so these are the eigenvalues of* B_n *and are therefore eigenvalues of* A'_n

$\square$

Future Directions

- SNF of B_n, the matrix of row sums of A_n, has invariant factors $(n-1)^k$
- Consider how permuting the columns of A_n affects eigenvalues
- What are the multiplicities of the eigenvalues?
  - Showing the multiplicities are what we suspect will show that these are the only eigenvalues of A_n
  - We suspect they are related to the sizes of the blocks of A_n when indexed correctly

"Nowhere in the sciences does one find as wide a gap as that between the written version of a mathematical result and the discourse that is required to understand the same result."- Gian-Carlo Rota

Acknowledgements

- Advisors Dr. Josh Ducey and Dr. Minah Oh
- REU director Dr. Len Van Wyk
- NSF Grant DMS-1004516
- JMU Department of Mathematics