Approaches to Rota's Basis Conjecture

Stephanie Bittner*

Virginia Wesleyan College

Xuyi Guo†

Stanford University

Adam Zweber‡

Carleton College

JMU Summer REU 2012

1 Introduction

Rota's Basis Conjecture. Rota's Basis Conjecture ('89) states that given a set of n bases for an n-dimensional vector space, one can always make n disjoint bases each containing one vector from each of the original bases. This conjecture is stated for any finite dimensional vector space over any field. Basically, if you have the bases,

$$\{a_1, a_2, \ldots, a_n\}, \{b_1, b_2, \ldots, b_n\}, \ldots, \{k_1, k_2, \ldots, k_n\}$$

we can place them as rows of this array:

$$\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ k_1 & k_2 & \cdots & k_n \end{array}$$

Then the conjecture states that there is a way to independently permute the rows so that each of the resulting columns forms a basis.

*snbittner@vwc.edu

†xuyguo@stanford.edu

‡zwebera@carleton.edu

1.1 Known Results

Relationship to the Alon-Tarsi Conjecture. Huang and Rota [5], showed that the Alon-Tarsi Conjecture implies Rota's Basis Conjecture over fields of characteristic zero. Then Onn [6] provided a much more simple proof using The Colorful Determinantal Identity.

The Alon Tarsi Conjecture states that for Latin squares of **even** size n the number of even Latin squares of size n and the number of odd Latin squares of size n are different.

In order to understand this, it is necessary to know how to tell if a Latin square is even or odd. In a Latin square, each row and column can be viewed as a permutation. The sign of a Latin square is the product of the signs of these permutations. If the sign is a positive 1, we have an even Latin square and a negative 1 means an odd Latin square.

Provided is a small 3 x 3 Latin square:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{array}$$

For example, in the second row we can see that there are two inversions. Therefore that row has a positive one corresponding to it. We continue this for every row and column, and conclude that this is an even Latin square.

Known Results for The Alon-Tarsi Conjecture. The Alon-Tarsi Conjecture has been verified by computer for all even $n \leq 8$. As n increases the number of the Latin squares of that size gets incredibly large and very difficult to work with.

Drisko [3] showed that the Alon-Tarsi Conjecture is true for dimension $n = p + 1$, where p is an odd prime. Drisko's method is to consider an action of $S_n \oplus S_n \oplus S_n$ on the set of all Latin squares of size n. He shows that the action respects the sign of a Latin square and that only a small number of orbits will have non-zero size modulo p^3, allowing him to analyze only these orbits. This analysis reveals that the difference between the number of even and odd Latin squares of size $n = p + 1$ is non-zero mod p^3.

Glynn [4] proved that the number of even Latin squares of order p - 1 is not equal to the number of odd Latin squares of that order. Therefore, Rota's Basis Conjecture is true for a vector space of dimension p - 1 over any field of characterstic zero or p, besides ones that do not divide the number of even Latin squares by the number of odd Latin squares.

Rota's Basis Conjecture for Matroids. Wendy Chan [2] solved Rota's Basis Conjecture for $n = 3$ bases in a rank 3 matroid. She used the Basis Exchange Theorem and solved the conjecture using 3 cases.

2 The Incidence Matrix of Disjoint Transversals

2.1 Transversals

Recall that Rota's Basis Conjecture claims that given a set of n bases for an n-dimensional vector space, one can always make n disjoint bases each containing one vector from each of the original bases. We can look at the original set of bases as the rows of an $n \times n$ array. By giving each basis a different color, we can interpret Rota's Basis Conjecture as claiming that one can form n disjoint "rainbow" bases, "rainbow" indicating that the basis has each color appear exactly once among its constituent vectors.

$$\begin{array}{cccc} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \\ \vdots & \vdots & & \vdots \\ k_1 & k_2 & \cdots & k_n \end{array}$$

Then we will define a ***transversal*** as any rainbow set of vectors, whether they form a basis or not. There are thus n^n such transversals since we have n colors of bases and n vectors within each of those bases.

2.2 The Incidence Matrix

Consider the matrix A_n which has rows and columns indexed by the set of all transversals of dimension n, and has a 1 in the (i, j)-th spot if the transversal indexing row i and the transversal indexing column j are *disjoint*. If the transversals are not disjoint then the matrix has a 0. Recall the transversals are sets of vectors, so two transversals being *disjoint* simply means that those two sets are disjoint. Then A_n will always have dimension $n^n \times n^n$ since there exist n^n distinct transversals of dimension n.

$$\begin{array}{cc} & \begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & (a_2, b_1) & (a_2, b_2) \end{array} \\ \begin{array}{c} (a_1, b_1) \\ (a_1, b_2) \\ (a_2, b_1) \\ (a_2, b_2) \end{array} & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

A_n for $n = 2$

A_n is also the adjacency matrix for the graph for which the set of all transversals forms the set of vertices and two vertices are connected by an edge if the transversals corresponding to them are disjoint. Below are images of A_n for $n = 3$ and $n = 5$. Black pixels represent an entry with value 1 and white pixels represent entries of 0.

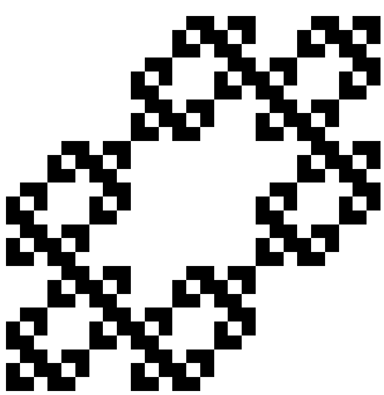

Figure 1: A_n for $n = 3$, a 27×27 matrix

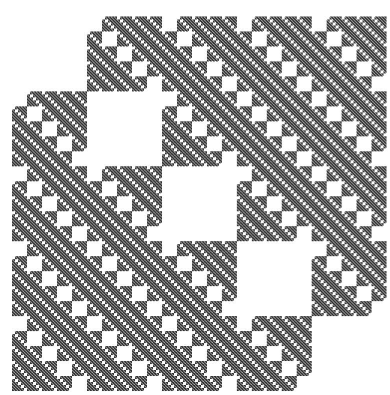

Figure 2: A_n for $n = 5$, a 3125×3125 matrix

It should be noted that the order in which we list the transversals when indexing this matrix is entirely arbitrary. Thus, these example images only represent what the incidence matrix will look like for a specific ordering of the transversals.

2.3 Why Consider A_n

We constructed the incidence matrix A_n in order to encapsulate information that would help us learn about Rota's Basis Conjecture. While not all transversals are bases, all bases are in fact transversals. Rota's Basis Conjecture requires finding n mutually disjoint rainbow bases, and disjointness of rainbow sets of vectors is what the matrix A_n tells us about. Additionally, A_n is interesting in itself as it can represent a more general combinatorial structure.

3 Smith normal form

3.1 Definition

Smith normal form is defined over any principle ideal domain, but for our purposes we consider it over the integers. Let X be any matrix over the integers. Then there exist unimodular (determinant=±1) integer matrices P and Q such that

$$PXQ = \begin{pmatrix} s_1 & 0 & 0 & & \cdots & 0 \\ 0 & s_2 & 0 & & \cdots & 0 \\ 0 & 0 & \ddots & & & 0 \\ & & & s_N & & \vdots \\ \vdots & & & & 0 & \vdots \\ 0 & & & \cdots & & 0 \end{pmatrix}$$

and s_i divides s_{i+1} for all $1 \le i \le N$. This matrix PXQ is called the Smith normal form of X. PXQ is diagonal and the diagonal entries s_i are called the *invariant factors*. Smith normal form exists for any matrix over the integers.

$$X = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 \\ 4 & -1 \end{pmatrix}, Q = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$SNF(X) = PXQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

Example of a non-square matrix put in Smith normal form

3.2 Smith normal form as an Equivalence Class Representative

From linear algebra over a field, such as the real numbers, we know that it is possible to multiply any matrix, X, on the left and right side by invertible square matrices, P and Q, such that the result is a matrix with some ones down the diagonal and zeroes everywhere else. We say this matrix is *equivalent* to the original matrix. This property defines equivalence classes that contain all matrices equivalent to exactly one such matrix with ones down the diagonal and zeros everywhere else.

$$PXQ = \begin{pmatrix} 1 & 0 & 0 & & & \cdots & 0 \\ 0 & 1 & 0 & & & \cdots & 0 \\ 0 & 0 & \ddots & & & & 0 \\ & & & & 1 & & \vdots \\ \vdots & & & & & 0 & \vdots \\ 0 & & & & \cdots & & 0 \end{pmatrix}$$

Smith normal form can be considered the counterpart to this over the integers, but yielding a matrix with the invariant factors instead of ones down the diagonal that we may call *integer equivalent* to a given integer matrix. It similarly forms an equivalence class, and as a result the Smith normal form of a matrix is unaffected by permuting the rows and columns of the matrix because that is essentially multiplying the matrix on the left and right side by permutation matrices. Permutation matrices are unimodular so the resulting product does not leave the equivalence class, in which there is a unique matrix that is the Smith normal form of all the integer equivalent matrices in that equivalence class. This fact is particularly useful since the order in which we put the transversals when indexing the incidence matrix for disjointness of transversals, A_n, is arbitrary, but when considering Smith normal form that doesn't matter, making Smith normal form a natural choice when trying to analyze A_n.

4 Conjecture About the Smith normal form of A_n

Smith normal form is very computationally complex, so it is difficult to calculate it for matrices as large as A_n. However, we calculated the Smith normal form of A_n for $n = 2, 3, 4, 5$. We now explain the notation we will be using in the following table. The first entry in the ordered pair denotes the value of the integer that appears in Smith normal form and the second entry denotes its multiplicity (i.e. $(1, 10)$ indicates that 1 appears on the Smith normal form diagonal 10 times.) Thus the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$ would correspond to $((1, 1), (3, 1))$

Dimension	Smith normal form
2	((1,4))
3	((1,8),(2,12),(4,6),(8,1))
4	((1,81),(3,108),(9,54),(27,12),(81,1))
5	((1,1024),(4,1280),(16,640),(64,160),(256,20),(1024,1))

The values of the invariant factors seem to have a pattern, that they are powers of $n - 1$, but the multiplicities seem somewhat more difficult. However, let us split the multiplicities for the 1s in dimension 2 and remake our table.

Dimension	Smith normal form
2	((1,1),(1,2),(1,1))
3	((1,8),(2,12),(4,6),(8,1))
4	((1,81),(3,108),(9,54),(27,12),(81,1))
5	((1,1024),(4,1280),(16,640),(64,160),(256,20),(1024,1))

Now a pattern in the multiplicities seems to be emerging. Interested in the fact that we have powers of $n-1$ let us multiply each value with its multiplicity (i.e. multiply together the entries of each ordered pair) and factor out $(n-1)^n$.

Dimension	
2	$(1^2\binom{2}{0}, 1^2\binom{2}{1}, 1^2\binom{2}{2})$
3	$(2^3\binom{3}{0}, 2^3\binom{3}{1}, 2^3\binom{3}{2}, 2^3\binom{3}{3})$
4	$(3^4\binom{4}{0}, 3^4\binom{4}{1}, 3^4\binom{4}{2}, 3^4\binom{4}{3}, 3^4\binom{4}{4})$
5	$(4^5\binom{5}{0}, 4^5\binom{5}{1}, 4^5\binom{5}{2}, 4^5\binom{5}{3}, 4^5\binom{5}{4}, 4^5\binom{5}{5})$

We can see that from each product we can actually factor out $(n-1)^n$ and then we are left with the entries of Pascal's Triangle, the binomial coefficients. Since we already saw a pattern in the values of the invariant factors, we are now ready to make our conjecture about what the Smith normal form for A_n will be.

4.1 Conjecture

The Smith normal form of A_n, which denotes the incidence matrix of disjoint transversals for dimension n, will have invariant factors with value

$$(n-1)^k$$

and multiplicity

$$(n-1)^{n-k}\binom{n}{k}$$

for $0 \leq k \leq n$.

5 Eigenvalues

5.1 Introduction

The data we collected on the incidence matrix of disjoint transversals for dimensions $n = 2, 3, 4, 5$ suggest that the eigenvalues of this matrix are, up to sign, the same as the invariant factors. This is a remarkable suggestion because little is known in general on the relationship between eigenvalues and invariant factors. Further, because A_n corresponds to the adjacency matrix of a graph, its eigenvalues are of interest to graph theorists. We were able to compute certain eigenvalues of a non-symmetric version of A_n. This matrix, which we will call A'_n is obtained by indexing of the rows and columns of A_n non-arbitrarily, then reversing the column order. As we will see, this reversal of column order simplifies eigenvalue computations without losing information about the incidence of any transversals. Additionally, permuting the columns of any matrix leaves the Smith normal form of that matrix unchanged, so it is relevant to our monomaniacal pursuit of the Smith normal form of A_n. However, permuting the columns of a matrix in general may change the eigenvalues, so we may need to account for this in future work. Despite these misgivings, we were able to show that $(n-1)^k$ is an eigenvalue of A'_n for each $k = 0, 1, \ldots, n$. We will need a couple lemmas to show this, but first a word on notation.

Let T_n be the set of n-dimensional transversals. Fix some $z \in T_n$ and denote by $T^k_{n,z}$ the set of transversals with exactly k elements in common with z. For example, if $n = 3$ and $z = (a_1, b_1, c_1)$ then $(a_2, b_2, c_2) \in T^0_{3,z}$, $(a_1, b_2, c_2) \in T^1_{3,z}$, $(a_1, b_1, c_2) \in T^2_{3,z}$ and $(a_1, b_1, c_1) \in T^3_{3,z}$. We use the $T^k_{n,z}$'s to index the rows and columns of A_n in increasing order from transversals in $T^0_{n,z}$ to transversals in $T^n_{n,z}$, where we order the elements *within* any $T^k_{n,z}$ in an arbitrary but fixed way. Thus we have

$$A_n = \begin{pmatrix} A_{0,0} & A_{0,1} & \cdots & A_{0,n} \\ A_{1,0} & A_{1,1} & \cdots & A_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,0} & A_{n,1} & \cdots & A_{n,n} \end{pmatrix},$$

where each $A_{i,j}$ is a block of A_n with rows indexed by elements of $T^i_{n,z}$ and columns indexed by $T^j_{n,z}$.

5.2 Row Sums

Lemma 1: Let a_{ij} denote the sum of all the elements in a row of $A_{i,j}$. Then $a_{ij} = (n-1)^i(n-2)^{(n-i-j)}\binom{n-i}{j}$.

Proof: Let $z_k \in T^i_{n,z}$, that is z_k indexes a row of $A_{i,j}$. Then, by the incidence relation, the row sum of row z_k in $A_{i,j}$ is equal to the number of transversals in $T^j_{n,z}$ that are disjoint from z_k. Now z_k has i entries in common with our original fixed transversal, z. This means that for any transversal disjoint to z_k, we have $n-1$ choices for these entries, yielding $(n-1)^i$ choices for such a transversal so far. For the remaining $n-i$ entries, we must have j entries in common with z. This yields $\binom{n-i}{j}$ choices. We now have $n-i-j$ entries to fill, each of which must be different from the corresponding entries in z and z_k. We know these entries of z and z_k are distinct because we have already accounted for those entries shared by z and z_k. Thus we have $(n-2)^{(n-i-j)}$ choices for these entries. Putting this all together, we have that the row sum of row z_k of $A_{i,j}$ is equal to $(n-1)^i(n-2)^{(n-i-j)}\binom{n-i}{j}$. This is independent of z_k so the row sums of each of these submatrices are constant. $\diamondsuit$

5.3 Finding Eigenvalues

Lemma 2 (Van Lint)[7]: Let M be a matrix of size m by m which has the form

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,k} \\ \vdots & \cdots & \ddots & \vdots \\ M_{k,1} & M_{k,2} & \cdots & M_{k,k} \end{pmatrix},$$

where each $M_{i,j}$ is a submatrix of size m_i by m_j $(i = 1, 2, \ldots, k; j = 1, 2, \ldots, k)$. Suppose that for each i and j the matrix $M_{i,j}$ has constant row sums b_{ij}. Let B be the matrix with entries b_{ij}. Then each eigenvalue of B is an eigenvalue of M.

Proof: Let $Bx = \lambda x$ where $x = (x_1, x_2, \ldots, x_k)$. Define y by

$$y = (x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2, \ldots, x_k, x_k, \ldots, x_k)$$

Where each x_i is repeated m_i times. Then by the definition of B, $My = \lambda y$. $\diamondsuit$

We now return to the eigenvalues of A_n.

Theorem: Let A_n be the incidence matrix of transversals of dimension n with rows and columns indexed as in lemma 1. Reverse the column order of A_n and denote this new asymmetric matrix by A'_n. Then for each $i \in \{0, 1 \ldots, n\}$, $(n-1)^i$ is an eigenvalue of A'_n.

Proof: Reverse the column order of A_n yielding the following block partition of A'_n

$$A'_n = \begin{pmatrix} A'_{0,n} & A'_{0,n-1} & \cdots & A'_{0,0} \\ A'_{1,n} & A'_{1,n-1} & \cdots & A'_{1,0} \\ \vdots & \vdots & \ddots & \vdots \\ A'_{n,n} & A'_{n,n-1} & \cdots & A'_{n,0} \end{pmatrix}.$$

Now by Lemma 1, each $A'_{i,j}$ corresponds to the submatrix $A_{i,j}$ above with reversed column order. Now in permuting the column order of a matrix, row sums are preserved so each $A'_{i,j}$ has constant row sum $(n-1)^i(n-2)^{(n-i-j)}\binom{n-i}{j}$ so Lemma 2 is applicable. Define the "matrix of row sums", B'_n corresponding to A'_n as in Lemma 2. Then the i,jth entry of B'_n now corresponds to block $A_{i,n-j}$. So b_{ij} is obtained by replacing j with $(n-j)$ in the formula above for the row sums the blocks of A_n. Thus $b_{ij} = (n-1)^i(n-2)^{(n-i-(n-j))}\binom{n-i}{n-j} = (n-1)^i(n-2)^{(j-i)}\binom{n-i}{n-j}$. Now if $i > j$ then $n-j > n-i$ so $\binom{n-i}{n-j} = 0$ so $b_{ij} = (n-1)^i(n-2)^{(n-i-j)}\binom{n-i}{j} = 0$. And if $i = j$ Then $b_{ij} = (n-1)^i(n-2)^{(j-i)}\binom{n-i}{n-j} = (n-1)^i(n-2)^0\binom{n-i}{n-i} = (n-1)^i$. Thus B is upper triangular with diagonal entries $(n-1)^i$, where i ranges from 0 to n. Thus these $(n-1)^i$'s are eigenvalues of B and therefore, by Lemma 2, eigenvalues of A'_n. $\diamondsuit$

6 Future Work

6.1 Effect of Column Permutations on Eigenvalues

By definition, an adjacency matrix is symmetric. This symmetry is a nice property that we could possibly exploit. For instance, symmetric matrices have orthonormal bases of eigenvectors. Given that it appears the eigenvalues of A_n are the invariant factors, it seems plausible that such an eigenbasis may not only be a basis for $\mathbf{R}^n$, but for $\mathbf{Z}^n$ as well. This would also simplify the proof of the Smith normal form by the fact that we would only need to show that one set of vectors is a basis of $\mathbf{Z}^n$. This is why we are currently investigating how permuting the columns of A_n changes the eigenvalues. One lemma that may help is due to Brouwer and Haemers[1]. It states that if A is an adjacency matrix for some graph Γ and P is a permutation matrix corresponding to an automorphism of Γ with order m, then the eigenvalues of AP are m-th roots of unity times the eigenvalues of A. In our case, our permutation matrix just reverses column order, which can be shown to never correspond to an automorphism of our graph corresponding to A_n. One could also imagine some permutation of the columns of A_n such that the resulting matrix still has the form of A'_n, only the columns *within* some submatrices $A_{i,j}$ have been permuted in a more complex way to make such a permutation an automorphism. This, as appealing as it may seem, also is probably impossible. So this lemma serves more as inspiration to investigate eigenvalues of the product of an adjacency matrix and a permutation matrix rather than a tool we can use directly.

6.2 Determinant of A_n

Determinants of Submatrices. Smith normal form can actually be calculated from the determinants of submatrices of a matrix. Let $d_i(X)$ denote the $i - th$ *determinant divisor* of matrix X, the greatest common divisor of the determinants of all $i \times i$ submatrices of X. Then

$$\prod_{i=1}^{k} s_i = d_k(X)$$

where the s_i denote the the invariant factors of X down the diagonal in Smith normal form.

From the fact about the relationship between the invariant factors of a matrix X and its determinant divisors, we decided to try to analyze the determinants of the submatrices of our matrix of interest, A_n. Unfortunately, this had little success as the determinants of the submatrices proved adequately difficult to work with. We then decided to just look at the determinant of the whole matrix A_n. In the case in which $n = p + 1$ where p is any prime, showing that the determinant is a power of p would be sufficient to show that all the invariant factors of A_n are powers of p as well. We were unable to find the determinant of A_n, but we were able to create a different problem which would give us the determinant. The determinant of a matrix is n-linear which means that we can express one column of A_n as the sum of two column vectors, call them $\vec{x}$ and $\vec{y}$, and the determinant of A_n will be equal to the sum of the determinants of the matrices formed by replacing the column vector with $\vec{x}$ and $\vec{y}$, respectively.

Notice that A_n will always have $(n-1)^n$ ones in each row and column, which is a result of there being $n-1$ vectors of a given color that are not in a particular transversal, and thus $(n-1)^n$ transversals which are disjoint to that transversal. A_n has n^n columns, so by repeatedly using the n-linearity of the determinant we can get that the determinant of A_n is equal to the sum of the determinants of $((n-1)^n)^{n^n} = (n-1)^{n^{n+1}}$ $n^n \times n^n$ matrices with only a single one in each column. Of these matrices, only the ones that also have a one in every row will have nonzero determinants since an entire row of zeros will give a zero determinant. However, a matrix that has a exactly a single one in every column and row can be interpreted as a permutation matrix. The determinants of the permuations matrices will always be ± 1 with the sign corresponding to the sign of the permutation. However, not all permutation matrices of dimension $n^n \times n^n$ can be created by decomposing A_n as described. Thus finding the determinant of A_n would be equivalent to finding the number of even and odd permutation matrices whose ones all correspond to spots with ones in A_n. This however, appears to be a difficult problem that we have yet to solve.

6.3 Powers of A_n

6.3.1 The Minimal Polynomial

The minimal polynomial of a square matrix X over a field is the polynomial P of least degree with leading coefficient 1 such that $P(X) = 0$. All other polynomials Q such that $Q(X) = 0$ will be multiples of the minimal polynomial, including the characteristic polynomial. The roots of the minimal polynomial are then all eigenvalues of the matrix X.

6.3.2 A_n^2

We wanted to find the minimal polynomial for A_n so we tried to express A_n^2 in terms of A_n and possibly some other matrices such as the identity matrix or the all-ones matrix. Whereas A_n indicates whether the transversal marking row i, call it t_i, and the transversal marking column j, call it t_j, are disjoint with a 1, A_n^2 indicates how many transversals are mutually disjoint to transversals t_i and t_j. We can think of these transversals in terms of how many elements they have in common, that is the size of the intersection of the two "rainbow" sets of vectors. Suppose that two transversals have k elements in common $0 \leq k \leq n$. One extreme is $k = 0$ in which case the transversals are actually disjoint and the other extreme is $k = n$ in which case the transversals are actually the same, just one is indexing the rows and one is indexing the columns. Then in all k spots that the transversals have in common there are $n - 1$ choices of possible vectors since we have only used 1 vector out of the n vectors of that color. In the other $n - k$ spots there are $n - 2$ choices of possible vectors the two transversals together have used 2 of the vectors of that color. Thus, if t_i and t_j have k elements in common then the (i, j)th entry in A_n^2 will be

$$(n-2)^{n-k}(n-1)^k.$$

Though we could not find a way to express A_n^2 in terms of A_n and the identity and all-ones matrix, we did find a way to describe it in terms of other matrices. Let $T_n^{(k)}$ be the incidence matrix of transversals with intersection of size k. This matrix has its rows and columns indexed by transversals, exactly like A_n, and has a 1 in the (i, j)th entry if $t_i \cap t_j = k$. Then we can see that we have the special cases $A_n = T_n^{(0)}$, since $t_i \cap t_j = 0$ means disjointness, and $T_n^{(n)}$ is the identity matrix if the rows and columns are indexed in the same order, since $t_i \cap t_j = n$ implies $i = j$. We can then express A_n^2 as

$$A_n^2 = \sum_{k=0}^{n} (n-2)^{n-k}(n-1)^k T_n^{(k)}.$$

6.4 Group Action

One idea to consider is to try to count the multiplicities of the invariant factors using a group action. To do this we use an action of S_{n^n} on two sets of vectors. The first set is the set of vectors of the form $A'_n z_i$ where A'_n is some matrix obtained by permuting the rows of A_n and z_i is a fixed column of A_n. Permuting the rows of A_n only changes the position of the entries in the vector $A_n z_i$, not the values, so $\mathrm{stab}(A_n z_i)$ is the set of all permutations which send entries of $A_n z_i$ to another entry with the same value. Now the entries of $A_n z_i$ are given by $\sum_{k=0}^{n} (n-2)^{n-k}(n-1)^k T_n^{(k)}$, so when this stabilizer acts on the set of indices $1, 2, \ldots, n^n$ the orbits are all of order $|T_n^{(k)}| = (n-1)^{n-k}\binom{n}{k}$. Now S_{n^n} acts on a different set of vectors, namely $(PA_nQ)'x$ where $(PA_nQ)'$ is some matrix obtained by a permutation of the rows of the SNF of A_n and x is the vector with every entry 1. Like in the action described above $\mathrm{stab}(PA_nQ)x$ must only send entries of $(PA_nQ)x$ to another entry with the same value. But the entries of $(PA_nQ)x$ are exactly the invariant factors of A_n including multiplicities. So when this stabilizer acts on the set of indices $1, 2, \ldots, n^n$, the orbits are exactly the multiplicities of the invariant factors of A_n. So showing that the stabilizers described above are isomorphic would show that the invariant factors each have multiplicity $(n-1)^{n-k}\binom{n}{k}$.

6.5 The case $n = p^2 + 1$

Some aspects of Drisko's method seem to lend promise to the case $n = p^2 + 1$. For example, Drisko's analysis of the orbits relies on there being a small number of neofields of size $n = p + 1$ because there are a small number of groups of order p, and every neofield of order n has an underlying group of order $n-1$. While for every prime p, there is only one group of order p, there are also only two groups of order p^2, meaning there are few neofields of order $n = p^2+1$, creating the impression of a possible way to extend Drisko's result. This is especially interesting in the unsolved case $n = 26$, because not only are there only two groups of order 25, but also only two groups of order 26, probably meaning there are very few neofields of this order. Now when trying to extend Drisko's method to the case $n = p^2+1$, the cause for concern is the question of counting modulo what. As Drisko shows, the order of the stabilizer of any Latin square must divide $n!n$, so when $n = p+1$, the order of the stablizer must either have 0 or 1 factors of p meaning the order of the orbit must have either p^3 or p^2 as a divisor because the order of $S_n \oplus S_n \oplus S_n$ is $(n!)^3 = p^3k$ for some constant k. This greatly simplifies matters because the order of any orbit of with p^3 as a divisor is going to be equal to 0 mod p^3, so Drisko needs only consider those orbits with p^2 as a divisor. In the case $n = p^2+1$ however, $n! = (p^2+1)! = p^{p+1}j$ where j is some constant. This means that even though the order of the stabilizer divides $n!n = (p^2+1)!n = p^{p+1}jn$, the order of the stabilizer may contain up to $p+1$ factors of p as opposed to 0 or 1 factors, so the matter is much more complicated than in Drisko's case.

7 Acknowledgments

We would like to thank our advisors Dr. Joshua Ducey and Dr. Minah Oh of James Madison University for guidance throughout our REU. We would also like to thank Dr. Leonard Van Wyk for directing the REU program, as well as NSF Grant DMS-1004516 for funding our research. Finally, we would like to thank Michael Cheung for producing helpful computational data.

References

[1] A. E. Brouwer and W. H. Haemers, *Spectra of graphs*, Universitext, Springer, New York, 2012.

[2] W. Chan, *An exchange property of matroid*, Discrete Math., 146 (1995), pp. 299–302.

[3] A. A. Drisko, *On the number of even and odd Latin squares of order* $p+1$, Adv. Math., 128 (1997), pp. 20–35.

[4] D. G. Glynn, *The conjectures of Alon-Tarsi and Rota in dimension prime minus one*, SIAM J. Discrete Math., 24 (2010), pp. 394–399.

[5] R. Huang and G.-C. Rota, *On the relations of various conjectures on Latin squares and straightening coefficients*, Discrete Math., 128 (1994), pp. 225–236.

[6] S. Onn, *A colorful determinantal identity, a conjecture of Rota, and Latin squares*, Amer. Math. Monthly, 104 (1997), pp. 156–159.

[7] J. H. van Lint, *Introduction to coding theory*, vol. 86 of Graduate Texts in Mathematics, Springer-Verlag, Berlin, third ed., 1999.