# Integer Invariants of Abelian Cayley Graphs

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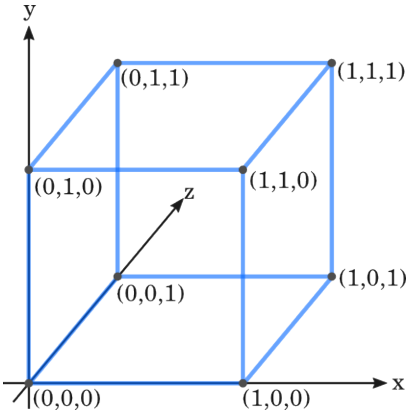
James Madison University

July 26, 2013

# Cube example.

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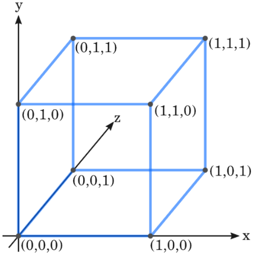


| | (0, 0, 0) | (0, 0, 1) | (0, 1, 0) | (0, 1, 1) | (1, 0, 0) | (1, 0, 1) | (1, 1, 0) | (1, 1, 1) |
|---|---|---|---|---|---|---|---|---|
| (0, 0, 0) | | | | | | | | |
| (0, 0, 1) | | | | | | | | |
| (0, 1, 0) | | | | | | | | |
| (0, 1, 1) | | | | | | | | |
| (1, 0, 0) | | | | | | | | |
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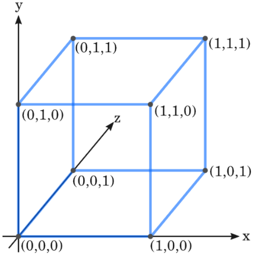


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|---|---|---|---|---|---|---|---|---|
| (0, 0, 0) | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| (0, 0, 1) | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 |
| (0, 1, 0) | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
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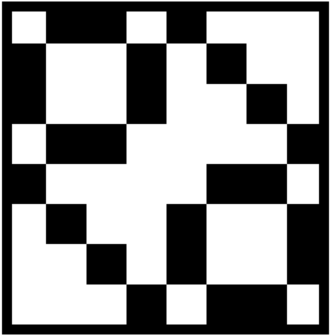

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$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

# Smith normal form.

- Given any integer matrix $A$, we can perform row and column operations so that:

$$A = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix} \text{ where } d_1|d_2 \text{ , } d_2|d_3, \text{ ... } ,d_{n-1}|d_n.$$

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- Swap any two rows/columns.
- Multiply any row/column by a nonzero integer.
- Add a integer multiple of one row/column to another.

# Cube example part 2.

| | (0,0,0) | (0,0,1) | (0,1,0) | (0,1,1) | (1,0,0) | (1,0,1) | (1,1,0) | (1,1,1) |
|---|---|---|---|---|---|---|---|---|
| (0,0,0) | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
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| (0,1,0) | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 |
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| (1,0,0) | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 |
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$$\begin{array}{c} \\ (0,0,0) \\ (0,0,1) \\ (0,1,0) \\ (0,1,1) \\ (1,0,0) \\ (1,0,1) \\ (1,1,0) \\ (1,1,1) \end{array} \begin{array}{c} \begin{array}{cccccccc} (0,0,0) & (0,0,1) & (0,1,0) & (0,1,1) & (1,0,0) & (1,0,1) & (1,1,0) & (1,1,1) \end{array} \\ \begin{pmatrix} 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \end{pmatrix} \end{array}$$

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|---|---|---|---|---|---|---|---|---|
| (0,0,0) | 3 | −1 | −1 | 0 | −1 | 0 | 0 | 0 |
| (0,0,1) | −1 | 3 | 0 | −1 | 0 | −1 | 0 | 0 |
| (0,1,0) | −1 | 0 | 3 | −1 | 0 | 0 | −1 | 0 |
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| (1,0,0) | −1 | 0 | 0 | 0 | 3 | −1 | −1 | 0 |
| (1,0,1) | 0 | −1 | 0 | 0 | −1 | 3 | 0 | −1 |
| (1,1,0) | 0 | 0 | −1 | 0 | −1 | 0 | 3 | −1 |
| (1,1,1) | 0 | 0 | 0 | −1 | 0 | −1 | −1 | 3 |

# Cube example part 2.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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- Invariant factors.

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- Invariant factors.
- Elementary divisors.

# Invariant factors vs. Elementary divisors

- $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 60 \end{pmatrix}$$

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- $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 \cdot 2 & 0 \\ 0 & 0 & 0 & 5 \cdot 3 \cdot 2^2 \end{pmatrix}$$

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- Invariant factors: $1, 3, 6, 60$
- $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 \cdot 2 & 0 \\ 0 & 0 & 0 & 5 \cdot 3 \cdot 2^2 \end{pmatrix}$$
- Elementary divisors: $2, 2^2, 3, 3, 3, 5$

# Back to the title.

- Integer

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- Integer
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- Abelian Cayley Graphs

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- Integer
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- Integer
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- Graph: edges and vertices
- If vertices come from finite abelian group $\Rightarrow$ abelian Cayley graph

# Group.

- Group: set of elements with an operation

$$\begin{array}{cc} (0,0,0) & (1,1,1) \\ (1,0,0) & (1,1,0) \\ (0,1,0) & (0,1,1) \\ (0,0,1) & (1,0,1) \end{array}$$

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- Closure
  Associativity
  Identity
  Inverses

# Connecting set $E$.

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- When is there an edge between two vertices?
- Define $E$.
- $E = \{(0,0,1),(0,1,0),(1,0,0)\}$
- Edge between $g$ and $h$ if $g - h$ is in $E$.

# Goals.

- Predict the elementary divisors and their multiplicities for various incidence matrices.

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- Recover the results of others in the field, but using a different technique.

# Relationship between eigenvalues and Smith normal form.

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- A number $\lambda$ is an eigenvalue of $A$ if there exists a nonzero vector $v$ such that $Av = \lambda v$.

# Relationship between eigenvalues and Smith normal form.

- $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

- $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

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- SNF: 2, 2, 2
- $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$
- Eigenvalues: 2, 2, 2
- SNF: 1, 1, 8

# Relationship between eigenvalues and Smith normal form.

- Theorem: For primes not dividing $|G|$, the multiplicity of $p^i$ as an elementary divisor of $A$, is the same as the number of eigenvalues exactly divisible by $p^i$. [Sin, 2012]

# Eigenvalues as character Sums.

- Theorem: $\frac{1}{|G|}MA\overline{M}^t = diag(\sum_{e \in E} \chi(e))$ where $\chi \in Irred(G)$ [MacWilliams-Mann, 1968]

# Hamming association scheme $H(n, q)$.

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- $G = Z_q \times Z_q \times \cdots \times Z_q$ ($n$ times)
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- The distance between $(0, 1, 2, 2, 1)$ and $(1, 1, 2, 2, 0)$ is 2.

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- Construct adjacency matrix $A_k$.

# Hamming association scheme $H(n, q)$

- Can find the $p$-elementary divisor multiplicities for primes $p$ not dividing $|G|$.

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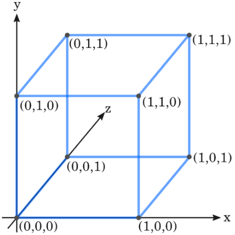
- 

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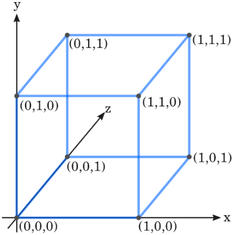
- 

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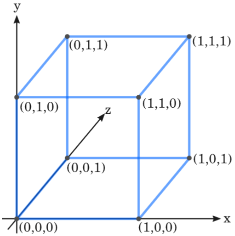
- 

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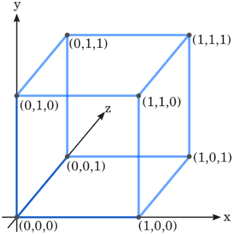
- 

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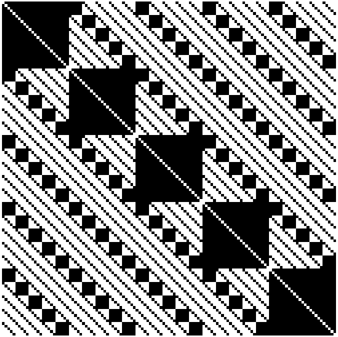
- H. Bai, 2002
  Jacobson-Niedermaier-Reiner, 2003

# Acknowledgements.


- Mentor Dr. Joshua Ducey
- Dr. Minah Oh
- Justin and Brock
- JMU Department of Mathematics and Statistics

# Questions?



$125 \times 125$ incidence matrix for the parameters $n = 3, q = 5$, and $k = 2$.