The Smith and Critical Group of the Rook's Graph and Its Complement

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Jonathan Gerhard*
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James Madison University

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Thank You!

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- the organizers of this symposium,
- Jeff Tickle for the funding to do this research!

The Rook's Graph

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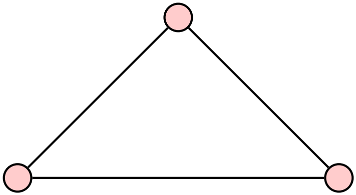

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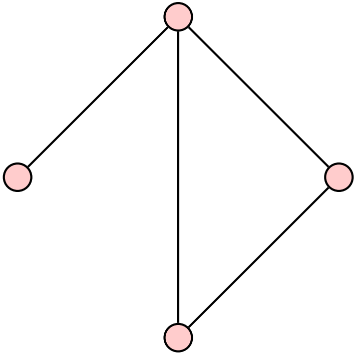

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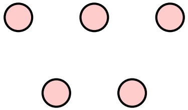

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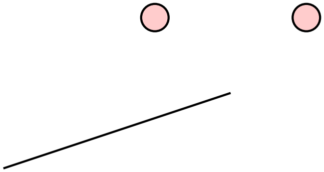

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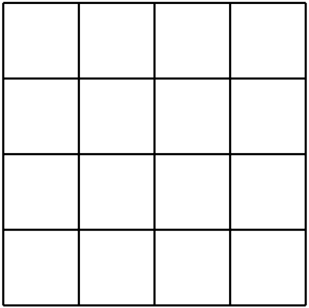

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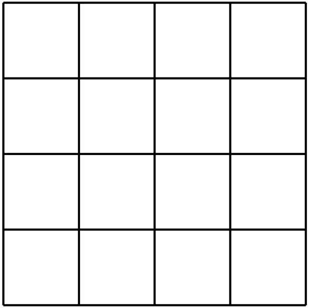

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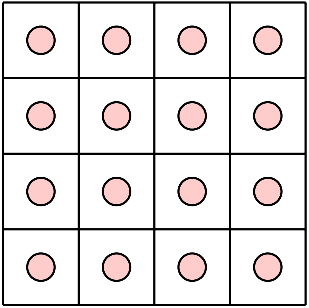

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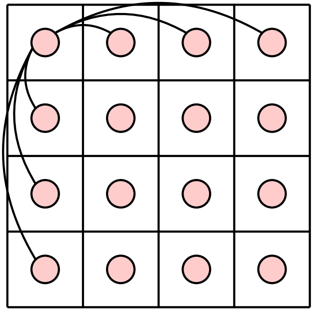
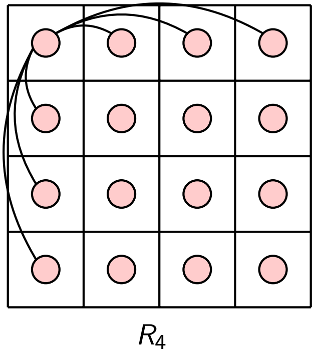
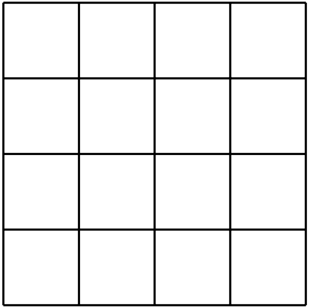



R_4

The Rook's Graph



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Graph Invariants

- A big question that arises in studying graphs is how do we know when two graphs are the same or different?

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- Sometimes it's obvious.
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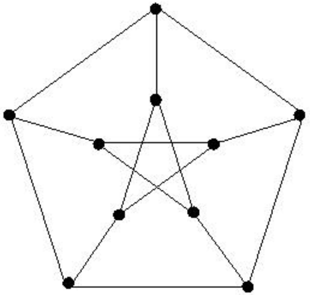

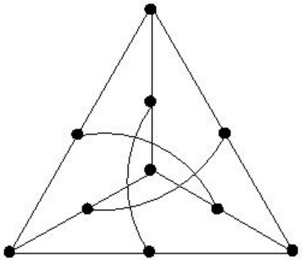

The Chip-Firing Game

- We begin with a graph and put some integer on each vertex, making sure the total sum is zero.

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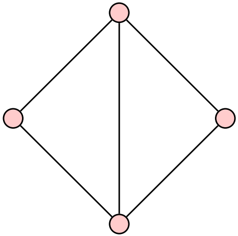

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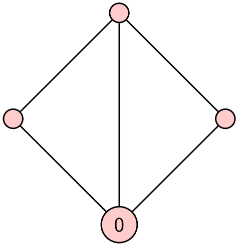

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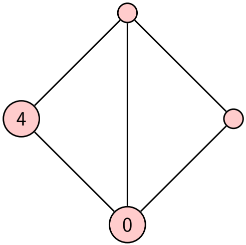

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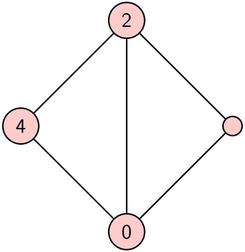

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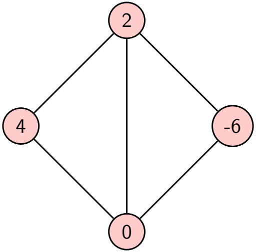

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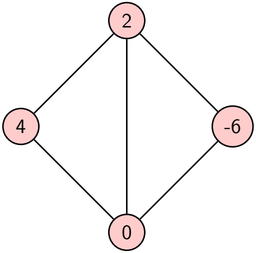
- Two configurations are equivalent if you can get from one to the other through chip-firing.



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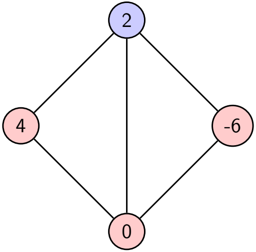
- When we fire a vertex, we decrease the number of chips on that vertex by its degree and send one chip to each of its neighbors.



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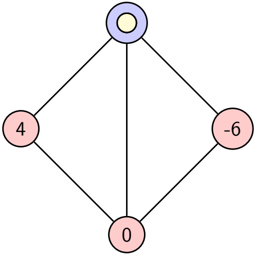

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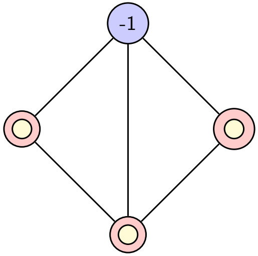

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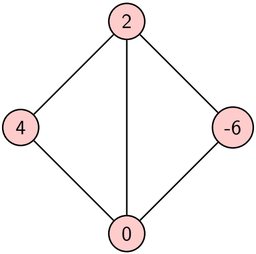
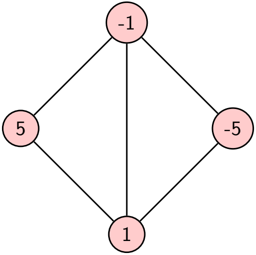
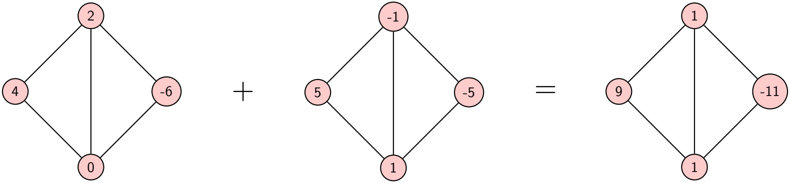




Adding Configurations



Lemma

Lemma

Let G be a finite abelian group, generated by the elements $x_1, x_2, \ldots, x_k$. Suppose that there exist integers $r_1, r_2, \ldots, r_k$ so that $|G| = r_1 \cdot r_2 \cdots r_k$ and $|x_i|$ divides r_i, for $1 \leq i \leq k$. Then

$$G \cong \mathbf{Z}_{r_1} \oplus \mathbf{Z}_{r_2} \oplus \cdots \oplus \mathbf{Z}_{r_k}.$$

To apply the lemma we need to know the order of our group, exhibit a set of elements and show that their orders divide the orders of the cyclic factors of the supposed decomposition, and show these elements do indeed generate the group.

Example

Lets consider R_4.

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$$\mathcal{K}(R_4) \cong (\mathbf{Z}_8)^5 \oplus (\mathbf{Z}_{32})^4 \, .$$

Set of Generators

-1	1		
1	-1		

-1	1		
1	-1		

-1		1	
1		-1	

-1		1	
1		-1	

-1	1		

-1		1	

-1			
1			

-1			
1			

-3	1	1	1

Orders

Here we take 8 times one of our generators for $\mathcal{K}(R_4)$ and show that it is equivalent to the all-zero configuration.

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-8	8		
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$\Rightarrow$

-2	7	-1	-1
7	-8		
-1			
-1			

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-2	7	-1	-1
7	-8		
-1			
-1			

$\Rightarrow$

-1	1		
7	-7		
-1	1		
-1	1		

$\Rightarrow$

-1			
6	-1	-1	-1
-1			
-1			

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-8	8		
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⇒

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-1			
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⇒

-1	1		
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-1	1		
-1	1		

⇒

-1			
6	-1	-1	-1
-1			
-1			

⇒

In the same way we can show that each of our elements have orders that divide the orders of the cyclic factors of the supposed decomposition.

Generating Set

-1	1		

-1		1	

-1			
1			

-1			
1			

Generating Set

To get the rest, we will show that they can be expressed as combinations of our elements or things that are equivalent to the all-zero configuration.

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-3	1	1	1

+

1	-1		

+

1		-1	

=

-1			1

Generating Set

To get the rest, we will show that they can be expressed as combinations of our elements or things that are equivalent to the all-zero configuration.

$$\begin{array}{|c|c|c|c|}\hline -3 & 1 & 1 & 1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline 1 & -1 & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline 1 & & -1 & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline -1 & & & 1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|}\hline -6 & 1 & 1 & 1 \\ \hline 1 & & & \\ \hline 1 & & & \\ \hline 1 & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline 3 & -1 & -1 & -1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & & & \\ \hline -1 & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline -1 & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline -1 & & & \\ \hline & & & \\ \hline & & & \\ \hline 1 & & & \\ \hline \end{array}$$

Generating Set

$$
\begin{array}{|c|c|c|c|}\hline 1 & & -1 & \\ \hline -1 & & 1 & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
+
\begin{array}{|c|c|c|c|}\hline -1 & & 1 & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
+
\begin{array}{|c|c|c|c|}\hline -1 & & & \\ \hline 1 & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
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$$

Generating Set

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\begin{array}{|c|c|c|c|}\hline 1 & & -1 & \\ \hline -1 & & 1 & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
+
\begin{array}{|c|c|c|c|}\hline -1 & & 1 & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
+
\begin{array}{|c|c|c|c|}\hline -1 & & & \\ \hline 1 & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
=
\begin{array}{|c|c|c|c|}\hline -1 & & & \\ \hline & & 1 & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
$$

$$
\begin{array}{|c|c|c|c|}\hline 1 & -6 & 1 & 1 \\ \hline & 1 & & \\ \hline & 1 & & \\ \hline & 1 & & \\ \hline \end{array}
+
\begin{array}{|c|c|c|c|}\hline 1 & & -1 & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
+
\begin{array}{|c|c|c|c|}\hline 1 & & & -1 \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
+
\begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & -1 & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
+
\begin{array}{|c|c|c|c|}\hline 1 & & & \\ \hline & & & \\ \hline & -1 & & \\ \hline & & & \\ \hline \end{array}
+
\begin{array}{|c|c|c|c|}\hline -6 & 6 & & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}
=
\begin{array}{|c|c|c|c|}\hline -1 & & & \\ \hline & & & \\ \hline & & & \\ \hline & 1 & & \\ \hline \end{array}
$$

Generating Set

$$\begin{array}{|c|c|c|c|}\hline 1 & & -1 & \\ \hline -1 & & 1 & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline -1 & & 1 & \\ \hline & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} + \begin{array}{|c|c|c|c|}\hline -1 & & & \\ \hline 1 & & & \\ \hline & & & \\ \hline & & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|}\hline -1 & & & \\ \hline & & 1 & \\ \hline & & & \\ \hline & & & \\ \hline \end{array}$$

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Since our set of elements do indeed generate the group, we know that $\mathcal{K}(R_4) \cong (\mathbf{Z}_8)^5 \oplus (\mathbf{Z}_{32})^4$

Main Theorem

Theorem

The critical group and Smith group of R_n and its complement R_n^c are given by the following isomorphisms:

$\mathcal{K}(R_n) \cong (\mathbf{Z}_{2n})^{(n-2)^2+1} \oplus (\mathbf{Z}_{2n^2})^{2(n-2)}$

$S(R_n) \cong (\mathbf{Z}_2)^{(n-2)^2} \oplus \left(\mathbf{Z}_{2(n-2)}\right)^{2n-3} \oplus \mathbf{Z}_{2(n-1)(n-2)}$

$\mathcal{K}(R_n^c) \cong \left(\mathbf{Z}_{n(n-2)}\right)^{(n-2)^2-1} \oplus \left(\mathbf{Z}_{n(n-1)(n-2)}\right)^2 \oplus \left(\mathbf{Z}_{n^2(n-1)(n-2)}\right)^{2(n-2)}$

$S(R_n^c) \cong \left(\mathbf{Z}_{(n-1)}\right)^{2(n-1)} \oplus \mathbf{Z}_{(n-1)^2}.$

Thank You!