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Summer 2019

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1 Dual Graphs

1.1 The Critical Group

In this section, we consider a **graph** G to be a finite set of **vertices** V_G together with a finite set of undirected **edges** E_G connecting them (we will drop subscripts if clear from context). We allow for multiple edges between a given pair of vertices, but not for connections from a vertex to itself ("self-loops"). We define E_G° to be the set of **oriented edges** of the graph, which includes two directed edges associated with each of the undirected edges in G.

For any $e \in E^\circ$, let $\overline{e} \in E^\circ$ denote the directed edge associated with the same undirected edge as e but with opposite orientation, and let $v_e^+, v_e^- \in V$ denote the vertices at the head and tail of e respectively. We then let $\mathbb{Z}E^\circ$ be the free $\mathbb{Z}$-module generated by the elements of E° mod the relation $\overline{e} = -e$, and $\mathbb{Z}V$ be the free $\mathbb{Z}$-module generated by the set of vertices V.

We can define the linear map $D_{1,G} : \mathbb{Z}E^\circ \to \mathbb{Z}V$ which sends each directed edge to the difference of its endpoints: $D_1(e) = v_e^+ - v_e^-$. We can also define its adjoint, $D_{1,G}^* : \mathbb{Z}V \to \mathbb{Z}E^\circ$, which operates on vertices by

$$D_1^*(v) = \sum_{\substack{e \in E^\circ \\ v = v_e^+}} e.$$

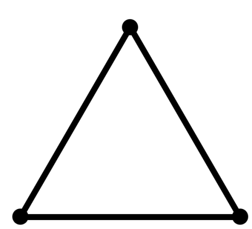
These are commonly referred to as the **edge-vertex incidence matrices** for G.

Now, consider the **Laplacian** $L_G = D_1 D_1^* : \mathbb{Z}V \to \mathbb{Z}V$. We have

$$L(v) = D_1(D_1^*(v)) = D_1\left(\sum_{\substack{e \in E^\circ \\ v = v_e^+}} e\right) = \sum_{\substack{e \in E^\circ \\ v = v_e^+}} D_1(e) = \sum_{\substack{e \in E^\circ \\ v = v_e^+}} v_e^+ - v_e^- = \sum_{v' \sim v} v - v',$$

where $\sim$ denotes adjacency. We are interested in the cokernel of L, $\mathrm{Coker}(L) = \mathrm{Codomain}(L)/\mathrm{Im}(L) = \mathbb{Z}V/L(\mathbb{Z}V)$. The rank of L is equal to $|V| - n$ where n is the number of connected components of G, so $\mathrm{Coker}(L) = \mathbb{Z}^n \oplus \kappa(G)$, where $\kappa(G)$ is a finite abelian group called the **critical group** of G.

For example, for the cycle graph C_3,



the edge-vertex incidence matrix D_1 is

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

and the Laplacian L is

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

The critical group is then $\mathrm{Coker}(L)/\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z}$.

1.2 Dual Graphs

A **2-cell embedding** $\Sigma : G \to M$ of a graph G into a 2-manifold M is an embedding such that the edges of G partition M into a set of **faces** $F_{G,\Sigma}$ that are each homeomorphic to a disk. For any such embedding Σ, there is an associated **dual graph** to G, denoted G^*_Σ (and an associated class of embeddings $\Sigma^* : G^*_\Sigma \to M$), a member of which is constructed by placing a vertex on each face in $\Sigma(G)$, and crossing each edge of $\Sigma(G)$ with an edge connecting the vertices of G^*_Σ associated to the faces of $\Sigma(G)$ incident on that edge. Note that this operation satisfies $(G^*_\Sigma)^*_{\Sigma^*} = G$.

We define the **face-edge incidence matrices** $D_{2,G,\Sigma}$ and $D^*_{2,G,\Sigma}$ of G as D^*_{1,G^*_Σ} and D_{1,G^*_Σ} respectively. Since there is a canonical bijection between $F_{G,\Sigma}$ and $V_{G^*_\Sigma}$, and between V_G and $F_{G^*_\Sigma,\Sigma^*}$, we will consider $D_{2,G,\Sigma}$ as a map from $\mathbb{Z}F_{G,\Sigma}$ to $\mathbb{Z}E^\circ_G$. $D^*_{2,G,\Sigma} : \mathbb{Z}E^\circ_G \to \mathbb{Z}F_{G,\Sigma}$, of course, remains its adjoint. $D_{2,G,\Sigma}$ sends faces of G to the (counterclockwise) sum of their surrounding edges and $D^*_{2,G,\Sigma}$ sends edges to the difference of the faces they border. The face-edge incidence matrices of G^* are precisely the vertex-edge incidence matrices of G.

Because the image of D_2 (for both a graph and its dual) is a sum of cycles, and sum of the differences of the endpoints of edges in a cycle is zero, we have $D_1 D_2 = 0$. This means that $\mathbb{Z}F$, $\mathbb{Z}E^\circ$, and $\mathbb{Z}V$ form a chain complex for both the graph and its dual:

$$\mathbb{Z}F_{G,\Sigma} \xrightarrow{D_{2,G}} \mathbb{Z}E^\circ_G \xrightarrow{D_{1,G}} \mathbb{Z}V_G$$

and

$$\mathbb{Z}F_{G^*} \xrightarrow{D_{2,G^*}} \mathbb{Z}E^\circ_{G^*} \xrightarrow{D_{1,G^*}} \mathbb{Z}V_{G^*}$$

We will now prove the isomorphism $\kappa(G) \cong \kappa(G^*)$ for graphs embedded into the 2-sphere.

First, note that for a graph with n connected components, the first homology group $H^0 = \mathbb{Z}V/\mathrm{Im}(D_1) \cong \mathbb{Z}^n$, so $\mathbb{Z}V/\mathbb{Z}^n \cong \mathrm{Im}(D_1)$. This gives us

$$\kappa(G) = \mathbb{Z}V/(\mathbb{Z}^n \oplus \mathrm{Im}(D_1 D_1^*) \cong \mathrm{Im}(D_1)/\mathrm{Im}(D_1 D_1^*).$$

Next, consider the group $\lambda(G) = \mathbb{Z}E^\circ/(\mathrm{Im}(D_2) \oplus \mathrm{Im}(D_1^*))$. The map $D_1 : \mathbb{Z}E^\circ \to \mathbb{Z}V$ induces a homomorphism

$$\mathbb{Z}E^\circ/(\mathrm{Im}(D_2) \oplus \mathrm{Im}(D_1^*)) \xrightarrow{D_1'} \mathrm{Im}(D_1)/\mathrm{Im}(D_1 D_1^*).$$

Since $D_1 D_2 = 0$, we have $D_1(\mathrm{Im}(D_2) \oplus \mathrm{Im}(D_1^*)) = \mathrm{Im}(D_1 D_1^*)$, so on the left side, we have only modded out things modded out on the right side; this means that the surjectivity of $\mathbb{Z}E^\circ \xrightarrow{D_1} \mathrm{Im}(D_1)$ makes D_1' surjective as well. To show injectivity, note that if $e \in \mathbb{Z}E^\circ$ satisfies $D_1(e) \in \mathrm{Im}(D_1 D_1^*)$, we must have $e = D_1^*(v) + e'$ for some $v \in \mathbb{Z}V$ and $e' \in \mathrm{Ker}(D_1)$.

If $\mathrm{Ker}(D_1) = \mathrm{Im}(D_2)$, then $e \in \mathrm{Im}(D_2) \oplus \mathrm{Im}(D_1^*)$ and so D_1' has trivial kernel; this means

$$\mathbb{Z}E^\circ/(\mathrm{Im}(D_2) \oplus \mathrm{Im}(D_1^*)) \cong \mathrm{Im}(D_1)/\mathrm{Im}(D_1 D_1^*) \cong \kappa(G),$$

and since

$$\lambda(G) = \mathbb{Z}E_G^\circ/(\mathrm{Im}(D_{2,G}) \oplus \mathrm{Im}(D_{1,G}^*)) \cong \mathbb{Z}E_{G^*}^\circ/(\mathrm{Im}(D_{1,G^*}^*) \oplus \mathrm{Im}(D_{2,G^*})) = \lambda(G^*),$$

the critical group of G is isomorphic to the critical group of its dual.

Our proof only works under the assumption that $\mathrm{Ker}(D_1) = \mathrm{Im}(D_2)$. Since $\mathrm{Ker}(D_1)$ is the free abelian group on cycles of edges, and $\mathrm{Im}(D_2)$ is the free abelian group on boundaries of faces, we see that the condition we need holds if and only if the face boundaries are a full basis for the cycles. This is the case when G is embedded into a 2-sphere, which has trivial first homology group.

1.3 Graphs on Surfaces of Nonzero Genus

We now investigate the case in which this is not true. The existence of cycles that do not bound a sum of face boundaries corresponds to the graph being 2-cell embedded into a surface of genus $g > 0$. We will assume all embeddings are 2-cell; if they are not, though it might seem that we would recover the 'all cycles bound faces' property, we also find that the dual of the dual graph is no longer isomorphic to the original graph.

The **genus** of a graph is the minimum genus of surface it has a planar embedding into. The **maximal genus** of a graph is the maximal genus of surface it can be 2-cell embedded into. We wanted to investigate the critical group and dual critical group of a planar graph on a surface of $g > 0$. Starting with the $g = 1$ case, we tried embedding graphs like K_4 into the torus:

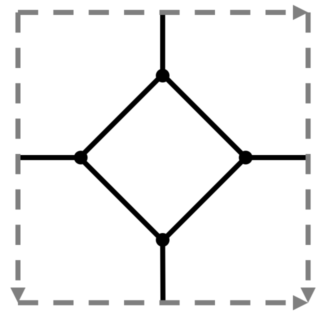


Figure 1: 2-cell embedding of K_4 on the torus

Though both faces in this embedding are homeomorphic to disks, and as such the embedding meets the conventional definition of being 2-cell, the 'outer' face is incident on itself, creating self-loops in the dual graph. To avoid this, we modified the graph to

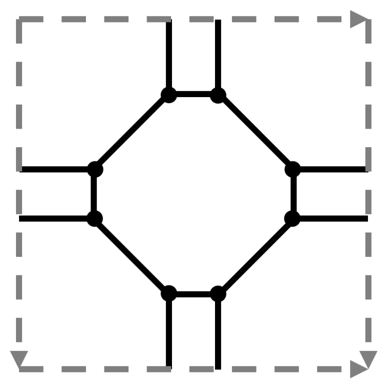


Figure 2: 2-cell embedding of Q_3 on the torus

which is just the 3-cube Q_3. As such, its critical group, and the critical group of its planar dual, is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/24\mathbb{Z}$. However, the critical group of its dual on the torus is $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/24\mathbb{Z}$, showing that the isomorphism does indeed break. We explored the results of applying the explicit isomorphism between $\kappa(G)$ and $\kappa(G^*)$ constructed in the referenced paper by Cori and Rossin to Q_3 on the torus, focusing on aspects such as the images of the generators of the group, but there was no clear pattern to the mapping.

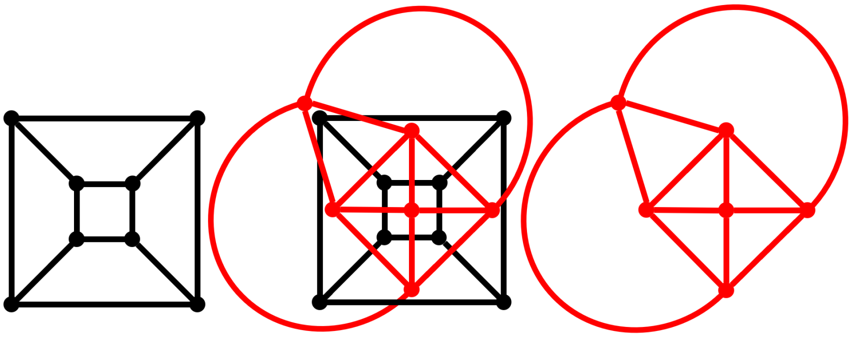


Figure 3: Q_3 and its dual on the 2-sphere

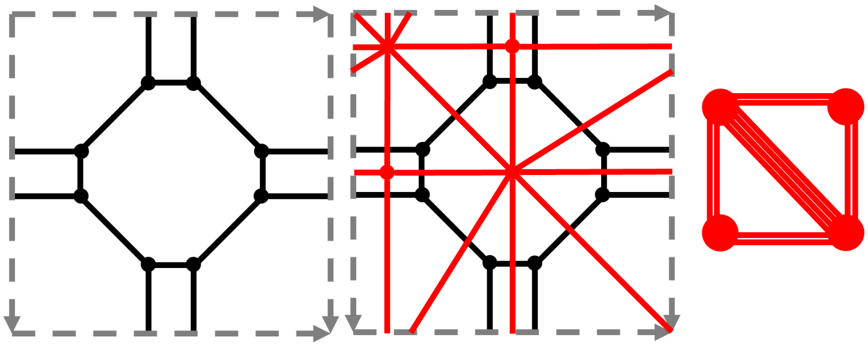


Figure 4: Q_3 and its dual on the torus

We next generalized this setup to surfaces of arbitrary genus: just as the torus can be represented as a square with edges identified, a surface of genus g can be represented as a $4g$-gon with edges identified as follows:

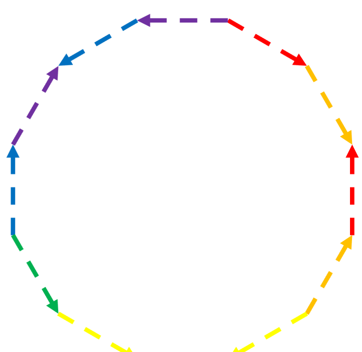


Figure 5: Genus 3 surface, with colors denoting pairs of edges to identify

We can do the same thing to this $4g$-gon as we did to the square:

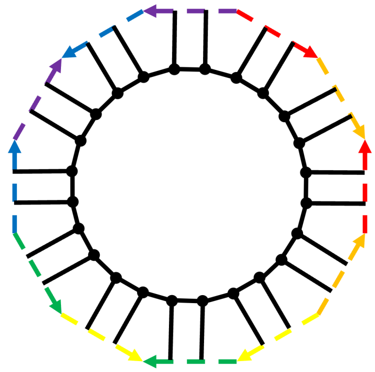


Figure 6: Q_3^3 on genus 3 surface

so that each 'hole' in the surface is '2-cell cut' by each subsequent set of 8 vertices. Since these 8-vertex subgraphs are planar and just connected in a circle, it turns out that this generalized "multi-cube" graph is planar for any g.

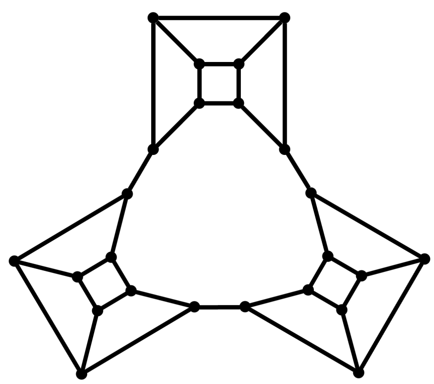


Figure 7: Q_3^3 on the 2-sphere

We will denote these "multi-cubes" by Q_3^g where g is their maximal genus. We conjecture (and experimentally verified up to $g = 25$) that the critical groups of these graphs are

$$\kappa(Q_3^g) \cong \mathbb{Z}_2^g \times \mathbb{Z}_8^{\min(2,g)} \times \mathbb{Z}_{80}^{\max(0,g-2)} \times \mathbb{Z}_{(24g)(10^{\min(1,\ g\text{-}1)})},$$

and the critical groups of their duals when embedded into a surface of genus g are

$$\kappa(Q_3^{g*}) \cong \mathbb{Z}_2^2 \times \mathbb{Z}_4^{2g-2} \times \mathbb{Z}_{24g}.$$

1.4 Different Embeddings

A basic fact about dual graphs is that the same graph can have different embeddings that produce non-isomorphic dual graphs. When embedding a graph into a surface of genus $g = 0$, we know that each of these non-isomorphic dual graphs must all have the same critical group; however, this is not true when $g > 0$; for example, the 6 different embeddings of K_5 into the torus

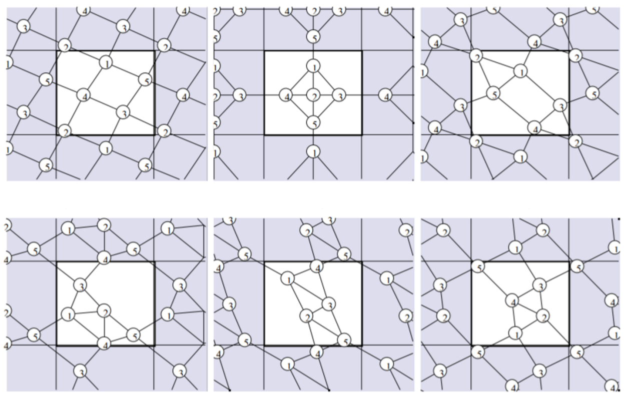


Figure 8: 2-cell embeddings of K_5 on the torus (image from Gagarin, Kocay and Neilson)

have duals with critical groups $\mathbb{Z}_5^3$ (the first embedding is self-dual), $\mathbb{Z}_3^2 \times \mathbb{Z}_5, \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7, \mathbb{Z}_3 \times \mathbb{Z}_5^2, \mathbb{Z}_5 \times \mathbb{Z}_{11}$, and $\mathbb{Z}_5 \times \mathbb{Z}_{19}$. However, if two embeddings differ in a 'planar' way–specifically, if they have the same $\text{Im}(D_2)$, we conjecture that the critical groups of the dual graphs will be the same.

2 Strongly-Regular Graphs

2.1 Co-Prime Eigenvalues

2.1.1 General Results on the Critical Group of SRG

Let Γ be a strongly regular graph with parameters (v, k, λ, μ). It is well known that the Laplacian matrix of Γ has three eigenvalues, 0 with multiplicity one (the *trivial eigenvalue*), r_L with multiplicity f and s_L with multiplicity g. We consider the case where $r_L, s_L \in \mathbb{N}$ with $\gcd(r_L, s_L) = 1$. In this instance, we write:

$$r_L = r_1^{a_1} \dots r_n^{a_n} \text{ and } s_L = s_1^{b_1} \dots s_m^{b_m}$$

where $r_1, \dots, r_n, s_1, \dots, s_m$ are distinct primes and $a_i, b_i \in \mathbb{Z}_{\geq 0}$ for all appropriate i.

Lemma 1. *The number of vertices v can be expressed in the form:*

$$v = r_1^{\gamma_1} \dots r_n^{\gamma_n} s_1^{\mu_1} \dots s_m^{\mu_m} \tag{1}$$

for some values $0 \leq \gamma_i \leq a_i f$ and $0 \leq \mu_i \leq b_i g$. Furthermore, the order of the critical group of Γ is:

$$|K(\Gamma)| = \frac{(r_1^{a_1} \dots r_n^{a_n})^f (s_1^{b_1} \dots s_m^{b_m})^g}{r_1^{\gamma_1} \dots r_n^{\gamma_n} s_1^{\mu_1} \dots s_m^{\mu_m}} = r_1^{a_1 f - \gamma_1} \dots r_n^{a_n f - \gamma_n} s_1^{b_1 g - \mu_1} \dots s_m^{b_m g - \mu_m}. \tag{2}$$

Proof. From Kirchhoff's matrix-tree Theorem, we have that:

$$K(\Gamma) = \frac{r_L^f s_L^g}{v} = \frac{(r_1^{a_1} \dots r_n^{a_n})^f (s_1^{b_1} \dots s_m^{b_m})^g}{v}.$$

Then for $K(\Gamma) \in \mathbb{N}$, it is clear that v must be as in (1). That $K(\Gamma)$ satisfies (2) is then easily verifiable. □

It is also a well known result that for strongly regular graphs, the elementary divisors of the Laplacian must divide $r_L \cdot s_L$. In our case, this means that:

$$\theta | r_1^{a_1} \dots r_n^{a_n} s_1^{b_1} \dots s_m^{b_m} \tag{3}$$

where θ denotes an elementary divisor of Γ.

2.1.2 Zeroing in on a Prime

In this section, we find the Sylow r_1-subgroup of $K(\Gamma)$, working over the ring $\mathbb{Z}_{(r_1)}$. From (3), we know that the critical group will take the form:

$$K_{r_1}(\Gamma) \cong (\mathbb{Z}/r_1\mathbb{Z})^{e_1} \oplus (\mathbb{Z}/r_1^2\mathbb{Z})^{e_2} \oplus \dots \oplus (\mathbb{Z}/r_1^{a_1}\mathbb{Z})^{e_{a_1}}.$$

We must find the values of $e_1, \dots, e_{a_1} \in \mathbb{N}$. One easily obtainable expression is:

$$e_0 + e_1 + \dots + e_{a_1} = v - 1 = f + g \tag{4}$$

where e_0 is the number of ones in the Smith Normal Form of the Laplacian of Γ over $\mathbb{Z}_{(r_1)}$ (called the r_1**-rank** of L over $\mathbb{Z}_{(r_1)}$). Brouwer informs us that, in our specific relatively-prime case, $e_0 = g$ or $g + 1$. Furthermore, $e_0 = g$ if and only if $r_1 | \mu$. In general (for our purposes) we will find that $e_0 = g$ or $g + 1$ if we are considering some r_i and $e_0 = f$ or $f + 1$ if we are considering an s_i, with an analogous criterion to determine which of these values is actually the r_i-rank of L. Hence, (4) reduces to:

$$e_1 + \cdots + e_{a_1} = f \text{ when } e_0 = g, \tag{5}$$

$$e_1 + \cdots + e_{a_1} = f - 1 \text{ when } e_0 = g + 1. \tag{6}$$

Additionally, we surmise from (2) that:

$$e_1 + 2e_2 + \cdots + (a_1 - 1)e_{a_1 - 1} + a_1 e_{a_1} = a_1 f - \gamma_1. \tag{7}$$

We need to find one more equation before we can proceed. This motivates the following definition. Let:

$$M_i = \{x \in \mathbb{Z}^n_{(p)} : p^i | Lx\}.$$

Then $M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots \supseteq M_{a_1} \supseteq \ker(L)$. Hence, $E_r \cap M_0 \subseteq M_{a_1}$. Finally, we have that:

$$f = \dim \overline{E_r \cap \mathbb{Z}^v_{(r_1)}} \leq \dim \overline{M_{a_1}} = e_{a_1} + 1,$$

or, stated more compactly that:

$$f - 1 \leq e_{a_1}. \tag{8}$$

Our next result yields an even sharper constraint on e_{a_1}.

Lemma 2. *The inequality $f - 1 \leq e_{a_1} \leq f$ is satisfied. Furthermore, $e_{a_1} = f$ if and only if $\gamma_1 = 0$.*

Proof. Assume $e_{a_1} > f$. Then:

$$e_1 + 2e_2 + \cdots + (a_1 - 1)e_{a_1 - 1} + a_1 e_{a_1} \geq a_1 e_{a_1} \geq a_1(f + 1) = a_1 f + a_1 > a_1 f - \gamma_1$$

which contradicts (7), so $f - 1 \leq e_{a_1} \leq f$ holds. We delay a proof of the second part of the Lemma, though we will eventually find that it is indeed true. □

We must now consider two cases, when $e_0 = g$ and when $e_0 = g + 1$. For the latter case, notice then that $e_{a_1} = f - 1$. For the former, $e_{a_1} = f$ or $e_{a_1} = f - 1$ are both possibilities. We now account for the fact that e_{a_1} must be at least $f - 1$, with the knowledge that $e_{a_1} = f$ is also possible when $e_0 = g$. Let $\overline{e_{a_1}} = e_{a_1} - (f - 1)$. Hence $\overline{e_{a_1}}$ equals zero or one. Notice then that (5) reduces to:

$$e_1 + \cdots + e_{a_1 - 1} + \overline{e_{a_1}} + (f - 1) = f \iff e_1 + \cdots + e_{a_1 - 1} + \overline{e_{a_1}} = 1.$$

Hence exactly one of $e_1, \ldots, e_{a_1-1}, \overline{e_{a_1}}$ is non-zero. Furthermore, substituting into (7) yields:

$$\begin{aligned}
&e_1 + 2e_2 + \cdots + (a_1-1)e_{a_1-1} + a_1\overline{e_{a_1}} + a_1(f-1) = a_1 f - \gamma_1 \\
\iff &e_1 + 2e_2 + \cdots + (a_1-1)e_{a_1-1} + a_1\overline{e_{a_1}} + a_1 f - a_1 = a_1 f - \gamma_1 \\
\iff &e_1 + 2e_2 + \cdots + (a_1-1)e_{a_1-1} + a_1\overline{e_{a_1}} = a_1 - \gamma_1.
\end{aligned}$$

Since exactly one of the e_i is non-zero, it must be that $e_{a_1-\gamma_1} = 1$ so that the identity is satisfied. On this note, we arrive at our main result for the section.

Theorem 1. *Let Γ, r_L, and s_L be stated above. Then, the r_1 part of the critical group $K(\Gamma)$ is:*

$$K_{r_1}(\Gamma) \cong (\mathbb{Z}/r_1^{a_1}\mathbb{Z})^{f-1} \oplus \mathbb{Z}/r_1^{a_1-\gamma_1}\mathbb{Z}.$$

Proof. For the most part, the above argument suffices. We saw that there were $f-1$ copies of e_{a_1} corresponding to $\mathbb{Z}/r_1^{a_1}\mathbb{Z}$, as well as one additional factor of $\mathbb{Z}/r_1^{a_1-\gamma_1}\mathbb{Z}$ corresponding to $e_{\alpha_1-\gamma_1}$. □

We conclude this section with a pair of corollaries, concerning applications of Theorem 1 to boundary cases.

Corollary 1. *When $\gamma_1 = 0$, we have that $K(\Gamma) \cong (\mathbb{Z}/r_1^{a_1}\mathbb{Z})^{f-1} \oplus \mathbb{Z}/r_1^{a_1}\mathbb{Z} = (\mathbb{Z}/r_1^{a_1}\mathbb{Z})^f$.*

Proof. In this case, the additional factor is $e_{a_1-0} = e_{a_1}$ (meaning that in the above expression $\overline{e_{a_1}} = 1$). □

The takeaway from Corollary 1 is that $e_{a_1} = f$ if and only if $\gamma_1 = 0$, as promised previously.

Corollary 2. *When $a_1 = 1$ and $\gamma \neq 0$, we have that $K(\Gamma) \cong (\mathbb{Z}/r_1\mathbb{Z})^{f-1} \oplus \mathbb{Z}/r_1^0\mathbb{Z} = (\mathbb{Z}/r_1\mathbb{Z})^{f-1}$*

2.1.3 Putting it all Together

Applying the techniques of Section 2.1.2 to each of our distinct primes, we arrive at our main result.

Theorem 2. *For any SRG with eigenvalues r_L and s_L as stated before, we have that:*

$$\begin{aligned}
K(\Gamma) \cong &(\mathbb{Z}/r_1^{a_1}\mathbb{Z})^{f-1} \oplus (\mathbb{Z}/r_1^{a_1-\gamma_1}\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/r_n^{a_n}\mathbb{Z})^{f-1} \oplus (\mathbb{Z}/r_n^{a_n-\gamma_n}\mathbb{Z}) \oplus \\
&(\mathbb{Z}/s_1^{b_1}\mathbb{Z})^{g-1} \oplus (\mathbb{Z}/s_1^{b_1-\mu_1}\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/s_m^{m_1}\mathbb{Z})^{g-1} \oplus (\mathbb{Z}/s_m^{b_m-\mu_m}\mathbb{Z})
\end{aligned}$$

Proof. Iteration of our procedure in Section 2.1.2 for each distinct prime yields the desired result. □

Notice that our result in Theorem 2 completely characterizes the critical group for SRG whose eigenvalues are relatively prime. Corollaries 1 and 2 have direct extensions to this more general case. We conclude this section with a pair of examples applying our theory.

Example 1. It is unknown if an SRG Γ with parameters (190,84,33,40) exists. However if such a graph exists, then its non-trivial eigenvalues are $r_L^f = 80^{133}$ and $s_L^g = 95^{56}$. Let's compute the Sylow 2-subgroup of $K(\Gamma)$. Notice that $2^4 || r_L$ whilst $2 \not| s_L$. Furthermore, $2^1||190$, meaning that $\gamma = 1$. Applying Theorem 1, we find that:

$$K_2(\Gamma) \cong (\mathbb{Z}/2^4\mathbb{Z})^{133-1} \oplus (\mathbb{Z}/2^{4-1}\mathbb{Z}) = (\mathbb{Z}/16\mathbb{Z})^{132} \oplus (\mathbb{Z}/8\mathbb{Z}).$$

Example 2. Conway's 99-graph problem posits the existence of an SRG Γ with parameters (99,14,1,2). It is known that its non-trivial eigenvalues of this hypothetical graph are $r_L^f = 11^{54}$ and $s_L = 18^{44}$. Notice then that $\gcd(r_L, s_L) = 1$. Hence we can apply Theorem 2 to obtain its critical group. Doing so, we find that:

$$K(\Gamma) \cong (\mathbb{Z}/11\mathbb{Z})^{53} \oplus (\mathbb{Z}/2\mathbb{Z})^{44} \oplus (\mathbb{Z}/9\mathbb{Z})^{43}$$

2.2 SRG with Common Factors

We now discuss SRG whose eigenvalues share common prime factors. Unfortunately, our characterization for this case is not as complete; we were only able to work out a couple of cases in detail.

2.2.1 Square-Free

First consider the case where r_L and s_L have a common prime factor, but $\gcd(r_L, s_L)$ does not contain a square. We consider the primes that r_L and s_L share, as our work for the distinct primes will be as before. Recycling notation from above, suppose $p||r_L$ (with multiplicity f) and $p||s_L$ (with multiplicity g) for p is prime. Then, by the Matrix-Tree Theorem:

$$|K_p(\Gamma)| = \frac{p^{f+g}}{p^\gamma} = p^{f+g-\gamma},$$

where $0 \le \gamma \le f+g$. As with above, we know that the invariant factors will divide $r_L s_L = p^2$. Hence:

$$K_p(\Gamma) \cong (\mathbb{Z}/p\mathbb{Z})^{e_1} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{e_2}.$$

We now have the expressions:

$$e_0 + e_1 + e_2 = f + g \tag{9}$$

$$\text{and } e_1 + 2e_2 = f + g - \gamma. \tag{10}$$

Unfortunately, we have not been able to a closed form solution for this system of equations (i.e. another equation or constraint that would enable us to solve for e_0, e_1, and e_2). However, we can find some conditions on the e_i. For instance, subtracting (10) from (9) and rearranging yields:

$$e_0 = e_2 + \gamma. \tag{11}$$

Furthermore, [1] informs us that $e_0 \leq \min\{f-1, g-1\}$. Fix $e_0 = n$. Then, from (11), $e_2 = n - \gamma$. Working with (9) informs us that:

$$e_1 = f + g + \gamma - 2n.$$

We thereby conclude that:

$$K_p(\Gamma) \cong (\mathbb{Z}/p\mathbb{Z})^{f+g+\gamma-2n} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{n-\gamma}. \tag{12}$$

Notice then that $f + g + \gamma - 2n \geq 0$ and $n - \gamma \geq 0$. As a minor result, we have slightly improved upon Brouwer's condition for e_0.

Corollary 3. *The inequality on the p-rank* $\gamma \leq e_0 \leq \frac{f+g+\gamma}{2}$ *holds.*

Example 3. An SRG Γ with parameters $(85, 27, 6, 9)$ is unknown to exist. The eigenvalues of this graph would necessarily be $r_L^f = 24^{55}$ and $s_L^g = 33^{32}$. Let's use (12) to study the Sylow 3-subgroup of $K(\Gamma)$. Fix $e_0 = n$. Notice that $\gamma = 0$ since $3^0||85$. Since $r_L = 3 \cdot 2^3$, and $s_L = 3 \cdot 11$, we find that:

$$K_3(\Gamma) \cong (\mathbb{Z}/3\mathbb{Z})^{55+32+0-2n} \oplus (\mathbb{Z}/3^2\mathbb{Z})^{n-0} = (\mathbb{Z}/3\mathbb{Z})^{87-2n} \oplus (\mathbb{Z}/9\mathbb{Z})^n.$$

2.2.2 p and p^2

For prime p, when $p||r_L$ and $p^2||s_L$, we obtain the following result:

$$K_p(\Gamma) \cong (\mathbb{Z}/p\mathbb{Z})^{f-n} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{g-n+\gamma} \oplus (\mathbb{Z}/p^3\mathbb{Z})^{n-\gamma},$$

or

$$K_p(\Gamma) \cong (\mathbb{Z}/p\mathbb{Z})^{f-n+1} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{g-n+\gamma-2} \oplus (\mathbb{Z}/p^3\mathbb{Z})^{n-\gamma+1},$$

where n is the p-rank of L, and γ is as before. That is, given the p-rank of L, there are only two options for $K_p(\Gamma)$. Similarly, when $p^2||r_L$ and $p||s_L$:

$$K_p(\Gamma) \cong (\mathbb{Z}/p\mathbb{Z})^{g-n} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{f-n+\gamma} \oplus (\mathbb{Z}/p^3\mathbb{Z})^{n-\gamma},$$

or

$$K_p(\Gamma) \cong (\mathbb{Z}/p\mathbb{Z})^{g-n+1} \oplus (\mathbb{Z}/p^2\mathbb{Z})^{f-n+\gamma-2} \oplus (\mathbb{Z}/p^3\mathbb{Z})^{n-\gamma+1}.$$

2.2.3 p^j and p^{j+1}

Let $r_L = p^j$ and $s_L = p^{j+1}$ for some $j \in \mathbb{N}$. Then:

$$|K_p(\Gamma)| = \frac{p^{jf+(j+1)g}}{p^\gamma} = p^{jf+(j+1)g-\gamma}.$$

Following the same process as before, we derive the identities:

$$e_0 + e_1 + \cdots + e_{2j} + e_{2j+1} = f + g,$$

$$e_1 + 2e_2 + \cdots + 2je_{2j} + (2j+1)e_{2j+1} = jf + (j+1)g - \gamma,$$

and using the other expressions (working with M_i and N_i), we find that:

$$f \leq e_0 + e_1 + \cdots + e_j \leq f + 1$$

and,

$$g - 1 \leq e_{j+1} + \cdots + e_{2j} + e_{2j+1} \leq g.$$

2.3 Table of SRG

A table of strongly regular graphs of up to 35 vertices from A. Brouwer's website, including their non-zero Laplacian eigenvalues, existence, and classification based on eigenvalue primeness (in accordance to the classification presented above), is present on the next page. Refer to the appendix for a longer table of SRG on less than 200 vertices.

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
5	2	0	1	1.38196601125011	3.61803398874989	Exists	Error
9	4	1	2	3	6	Exists	Square Free
10	3	0	1	2	5	Exists	Relatively prime
	6	3	4	5	8	Exists	Relatively prime
13	6	2	3	4.69722436226801	8.30277563773199	Exists	Error
15	6	1	3	5	9	Exists	Relatively prime
	8	4	4	6	10	Exists	Square Free
16	5	0	2	4	8	Exists	NA
	10	6	6	8	12	Exists	NA
16	6	2	2	4	8	Exists	NA
	9	4	6	8	12	Exists	NA
17	8	3	4	6.43844718719117	10.5615528128088	Exists	Error
21	10	3	6	9	14	Exists	Relatively prime
	10	5	4	7	12	Exists	Relatively prime
21	10	4	5	8.20871215252208	12.7912878474779	DNE	Error
25	8	3	2	5	10	Exists	Square Free
	16	9	12	15	20	Exists	Square Free
25	12	5	6	10	15	Exists	Square Free
26	10	3	4	8	13	Exists	Relatively prime
	15	8	9	13	18	Exists	Relatively prime
27	10	1	5	9	15	Exists	P, P2
	16	10	8	12	18	Exists	P, P2
28	9	0	4	8	14	DNE	P, P2
	18	12	10	14	20	DNE	P, P2
28	12	6	4	8	14	Exists	P, P2
	15	6	10	14	20	Exists	P, P2
29	14	6	7	11.8074175964327	17.1925824035673	Exists	Error
33	16	7	8	13.627718676731	19.372281323269	DNE	Error
35	16	6	8	14	20	Exists	P, P2
	18	9	9	15	21	Exists	Square Free

2.4 Latin Square Graphs

Conjecture: For a Latin Square graph, with n^2 vertices, that is constructed from the corresponding back-circulant Latin Square, the critical group is as follows when $n \geq 5$:

$$\kappa(\Gamma) \approx \begin{cases} (\mathbb{Z}/\frac{n}{2}\mathbb{Z}) \oplus (\mathbb{Z}/n\mathbb{Z})^2 \oplus (\mathbb{Z}/3n\mathbb{Z})^{n^2-6n+6} \oplus (\mathbb{Z}/6n\mathbb{Z})^2 \oplus (\mathbb{Z}/12n\mathbb{Z})^2 \oplus (\mathbb{Z}/6n^2\mathbb{Z})^{3n-8} & \text{n is even} \\ (\mathbb{Z}/n\mathbb{Z}) \oplus (\mathbb{Z}/3n\mathbb{Z})^{n^2-6n+5} \oplus (\mathbb{Z}/6n\mathbb{Z})^5 \oplus (\mathbb{Z}/6n^2\mathbb{Z})^{3n-8} & \text{n is odd} \end{cases}$$

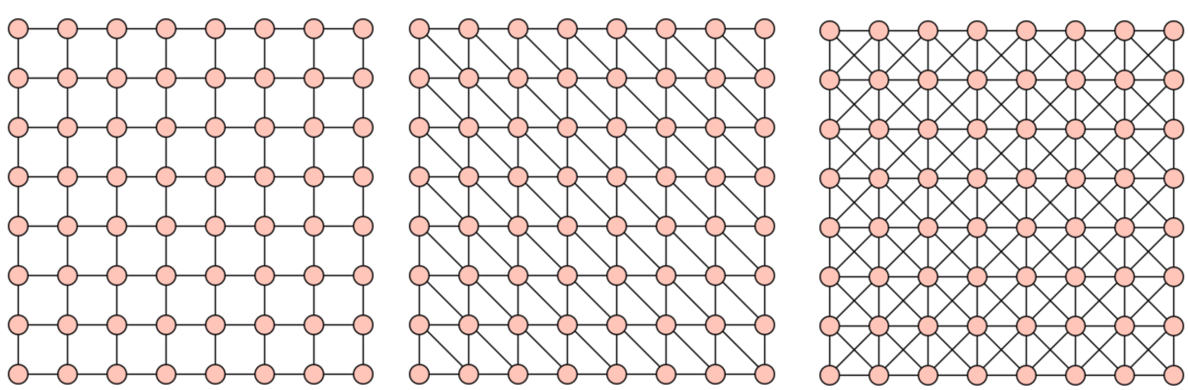
This was found through observation, but has not been proven.

3 Grid Graphs

The behavior of sandpiles on finite subgraphs of the $\mathbb{Z}^2$ lattice has been long studied. Primarily, interest has been in describing the identity of such graphs (typically on square or rectangular subsections of the lattice) and the result of stabilizing a configuration where a single vertex has a large number of chips and all other vertices has a constant number, less than their degree (a process we will refer to as single pile toppling).

Deepak Dahr conjectures that on a square grid, there is a square central region of dimensions proportional to that of the grid with two chips on each vertex and Yvan Le Borgne and Dominique Rossin describe the identity on a certain class of rectangular grids more fully in *On the identity of the sandpile group.* However, full descriptions and explanation of behaviors exhibited in such identities have yet to be made.

Here, we look at finite subgraphs based on three grid graphs as shown below and discuss some observed behavior.



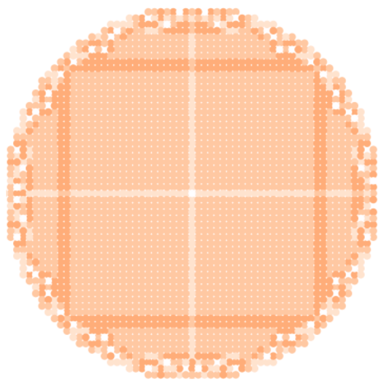
C_4 tiling Skewed triangle tiling K_4 tiling

For all examples shown, we approximate the sink at infinity by connecting it to all boundary vertices, i.e. those with non-maximal degree. Moreover, we allow multiedges to the sink so that all vertices of each graph has the same degree.

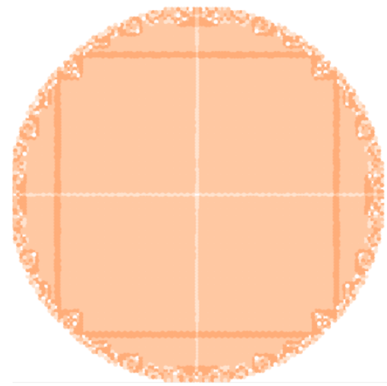
We focus primarily on the K_4 tiling though we will start with some results on the C_4 and skewed triangle tilings.

3.1 Graphs on the C_4 tiling

Graph on 60×60 vertices



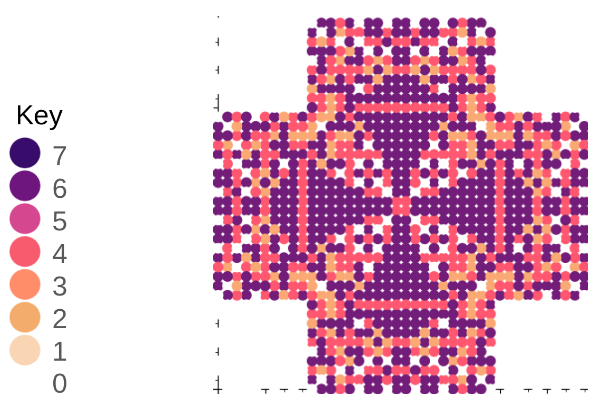
Graph on a circular graph of radius 30



Graph on a circular graph of radius 60

Darker orange represents more chips on the vertex (0 to 3 chips possible), with the central square regions of each graph having 2 chips per vertex.

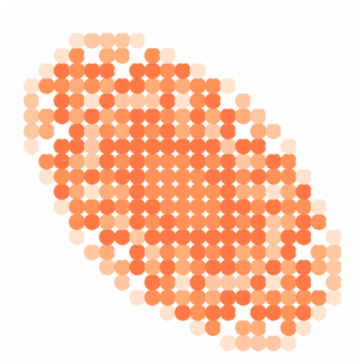
Like with the square, there is clear structure on the C_4 tiling with a circular boundary with the central region having 2 chips and a cross of vertices with 1 chip and square of vertices with 3 chips.



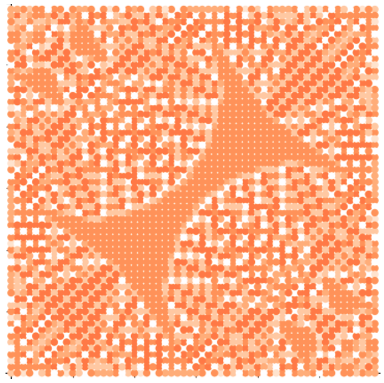
An example of a subgraph with a non-convex boundary

Note, there are vertices with 0 chips at each of the 8 corners

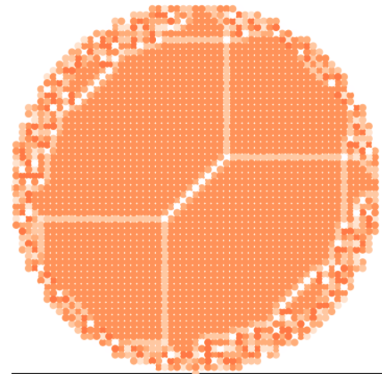
3.2 Graphs on the skewed triangle tiling



Single pile toppling with 1000 chips



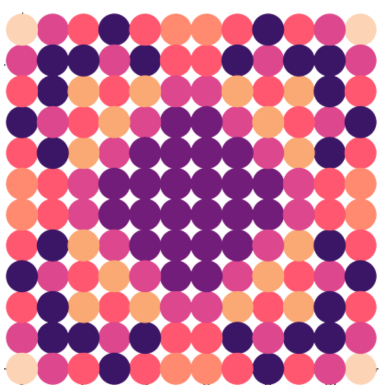
Identity on 60×60 vertices



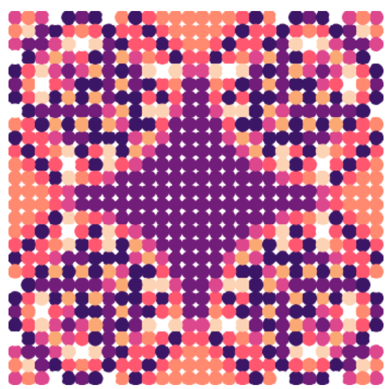
Identity on a circular graph of radius 30

We can see how the additional diagonals present results in the chips traveling further on one diagonal axis than the other as expected. The identity's central region is likewise stretched out with the central constant region having 4 chips per vertex.

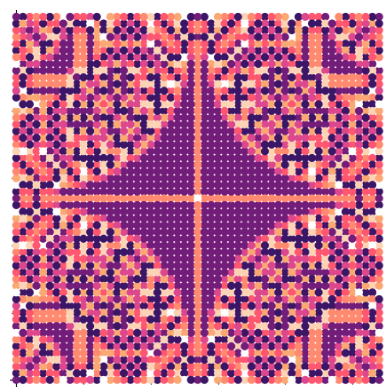
3.3 Graphs on the K_4 tiling



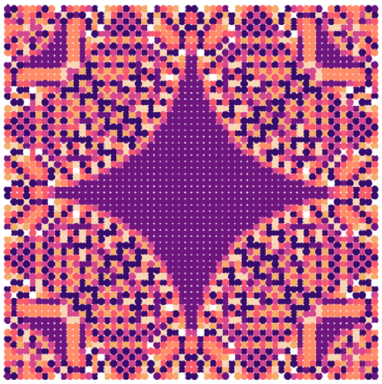
Identity on 12×12 vertices



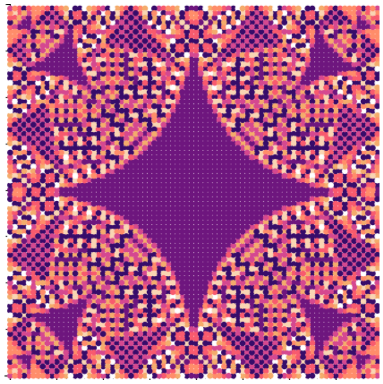
Identity on 28×28 vertices



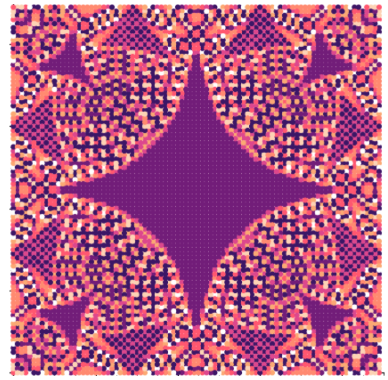
Identity on 55×55 vertices



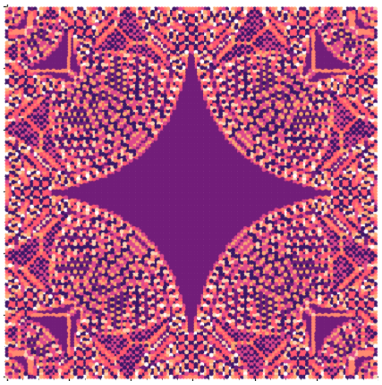
Identity on 60×60 vertices



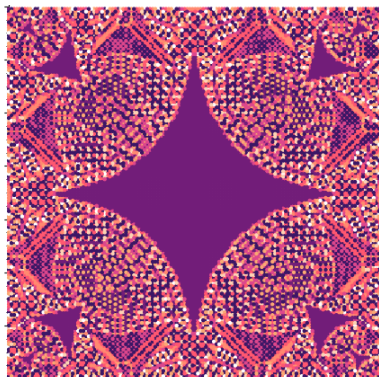
Identity on 80×80 vertices



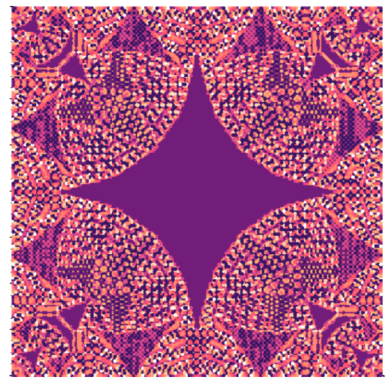
Identity on 90×90 vertices



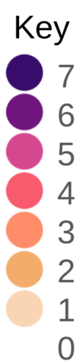
Identity on 120×120 vertices

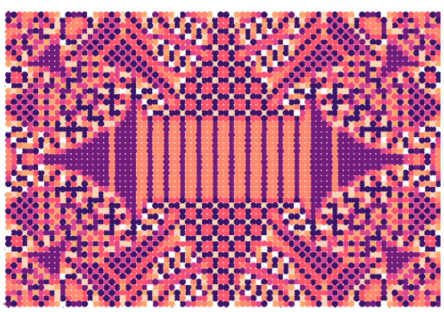


Identity on 140×140 vertices

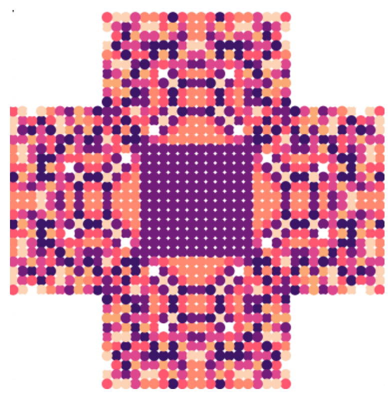


Identity on 160×160 vertices





Identity on 80×54 vertices



Identity with a non-convex boundary

We observe that on a square grid, these identities have a central region of vertices with 6 chips each, similar to that of the $C_4 tiling$. Here, however, it is a curving diamond shape. In addition, like with the C_4 tiling, grids of odd length also have a cross through the center with half the number of chips and the center vertex has 0 chips. We note that with the C_4, skewed triangle, and K_4 grids, the constant central region has $degree - 2$ chips on each vertex.

We can also observe that the length of the central region of the K_4 tiling grids are proportional to that of the overall graph (this has likewise been conjectured for the C_4 tiling grid) and there exists a fractal structure within the identities with patterns repeating in the corners. It is believed but unproven that if ever larger grids were normalized (as they are in these images so that they are of the same size), as the dimension goes to infinity, the central region would converge to a continuous region for which a boundary function could be found.

In addition, since K_4 can also be interpreted as a tetrahedral, this graph could be realized as stitched together tetrahedrals in 3-space (not space filing). Such a graph would have equal length edges, unlike this projection. As a result, this un-equal edge length projection has a strong impact on the resulting shape of the central region, though the impact of such an observation is unknown.

4 Indivisible Sandpiles

4.1 Sandpiles

Let Γ be a graph on n vertices with $n-1$ edges at every vertex. The Laplacian matrix L is an $n \times n$ matrix defined by

$$L = D - A$$

where A is the adjacency matrix and D is the incidence matrix. Assign a sink to a vertex on Γ and delete the corresponding row and column from the Laplacian to produce the reduced Laplacian, L', an $(n-1) \times (n-1)$ matrix

$$L' = \begin{pmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{pmatrix}_{(n-1)\times(n-1)}$$

Let the set of all nonsink vertices be V' such that $V' \cong \{v_1, v_2, \dots, v_{n-1}\}$. A stable sandpile is one such that the number of chips at each vertex $v \in V'$ is less than the degree of that vertex (in the case of the complete graph, $\sigma < n-1$ everywhere). An unstable sandpile on the graph is $\sigma \geq n-1$ for at least one vertex. A sandpile is stabilized by firing every $v \in V'$ until everywhere $\sigma < d-1$. The stabilized sandpile may be defined as τ. The number of times a vertex is fired to reach τ is denoted by $\tilde{u}$. Let $\tilde{u}$, d, and σ be vectors where each element corresponds to the value at one vertex.

4.2 Odometer

Definition 4.1. The odometer u is the minimum $u : V' \to \mathbb{Z}$ satisfying

$$\sigma - L'u \leq d - 1 \tag{13}$$

$$u \geq 0 \tag{14}$$

Theorem 3. *Let σ be a sandpile on any graph Γ and consider $\tilde{u} : V' \to \mathbb{Z}$ which satisfies*

$$\sigma - L'\tilde{u} = d - 1 \tag{15}$$

$$\tilde{u} \geq 0 \tag{16}$$

Then $\tilde{u}$ is the odometer.

Proof. We can rearrange (13) and (15) such that

$$L'\tilde{u} = \sigma - d + 1 \quad \text{and} \quad L'u \geq \sigma - d + 1$$

Then

$$L'u \geq L'\tilde{u}$$

$$L'u - L'\tilde{u} \geq 0$$

$$L'(u - \tilde{u}) \geq 0$$

Let $q = u - \tilde{u} \leq 0$ because u is the minimum, and

$$L'q \geq 0 \tag{17}$$

L' is a positive definite matrix and thus has the property that, for any vector x, the inner product $\langle x, L'x \rangle \geq 0$. Hence,

$$0 \leq \langle q, L'q \rangle = \sum_{v \in V'} q(v)(L'q)(v) \leq 0$$

Then, from the restrictions on either side, $\langle q, L'q\rangle = 0$ which is only true for $q = 0$. Therefore,

$$u = \tilde{u}$$

□

Definition 4.2. The d-odometer u_d is the minimum $u_d : V' \to \mathbb{R}$ satisfying

$$\sigma - L'u_d \leq d - 1 \tag{18}$$

$$u_d \geq 0 \tag{19}$$

Theorem 4. *Let σ be a sandpile on any graph Γ and consider some $\tilde{u} : V' \to \mathbb{R}$ which satisfies*

$$\sigma - L'\tilde{u} = d - 1$$

$$\tilde{u} \geq 0$$

Then $u_d = \tilde{u}$ is the d-odometer.

Proof. The same method we used for proving 3 applies here. □

4.3 Inverse Reduced Laplacian

The inverse reduced Laplacian $(L')^{-1}$ is a key element of classifying the indivisible sandpiles on different types of graph. We will discuss the properties of $(L')^{-1}$ for the complete graph, cycle graph, wheel graph, and path graph.

Let $E(\Gamma) \equiv$ *elementary divisors* of Γ and $E_{\max} \equiv \max(E(\Gamma))$.

4.3.1 Complete Graph

The general form of $(L')^{-1}$ for the complete graph K_n is

$$(L')^{-1} = \frac{1}{E_{\max}} \begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{pmatrix}$$

In the case of K_n, $E_{\max} = n$ such that

$$(L')^{-1} = \frac{1}{n}\begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{pmatrix} \tag{20}$$

4.3.2 Cycle Graph

The general form of $(L')^{-1}$ for the cycle graph on n vertices is

$$(L')^{-1} = \frac{1}{E_{\max}}\begin{pmatrix} n-1 & n-2 & n-3 & \dots & 3 & 2 & 1 \\ n-2 & 2(n-2) & 2(n-3) & \dots & 2\cdot 3 & 2\cdot 2 & 2\cdot 1 \\ n-3 & 3(n-2) & 3(n-3) & \dots & 3\cdot 3 & 3\cdot 2 & 3\cdot 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 3\cdot 1 & 3\cdot 2 & 3\cdot 3 & \dots & 3(n-3) & 2(n-2) & (n-2) \\ 2\cdot 1 & 2\cdot 2 & 2\cdot 3 & \dots & 2(n-3) & 2(n-2) & (n-2) \\ 1 & 2 & 3 & \dots & n-3 & n-2 & n-1 \end{pmatrix} \tag{21}$$

4.3.3 Wheel Graph

The general form of $(L')^{-1}$ for the cycle graph on n vertices is dependent on the Fibonacci numbers when n is even and the Lucas numbers when n is odd.

- Consider when n is even. The i^{th} number of Fibonacci sequence is F_i.

$$(L')^{-1} = \frac{1}{E_{\max}}\begin{pmatrix} F_n & F_{n-2} & F_{n-4} & \dots & F_2 & F_2 & \dots & F_{n-2} \\ F_{n-2} & F_n & F_{n-2} & \dots & F_2 & F_2 & \dots & F_{n-4} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ F_2 & F_4 & \dots & F_{n-2} & F_n & F_{n-2} & \dots & F_2 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ F_{n-2} & F_{n-4} & \dots & F_2 & F_2 & \dots & F_{n-2} & F_n \end{pmatrix}_{(n-1)\times(n-1)} \tag{22}$$

The sequence $F_n, F_{n-2}, \dots, F_4, F_2, F_2, F_4, \dots, F_{n-2}$ cycles through every row where, for the j^{th} row, F_n starts in the j^{th} column, such that $0 \leq j < n$.

- Consider when n is odd. The i^{th} number of the Lucas numbers is A_i.

$$(L')^{-1} = \frac{1}{E_{\max}} \begin{pmatrix} A_{n-1} & A_{n-3} & A_{n-5} & \dots & A_2 & \dots & A_{n-3} \\ A_{n-3} & A_{n-1} & A_{n-3} & \dots & A_2 & \dots & A_{n-5} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_2 & A_4 & \dots & A_{n-3} & A_{n-1} & A_{n-3} & \dots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n-3} & A_{n-5} & \dots & A_2 & \dots & A_{n-3} & A_{n-1} \end{pmatrix}_{(n-1)\times(n-1)} \tag{23}$$

The sequence $A_{n-1}, A_{n-3}, \dots, A_4, A_2, A_4, \dots, A_{n-3}$ cycles through every row where, for the j^{th} row, A_n starts in the j^{th} column, such that $0 \leq j < n$.

4.3.4 Path Graph

The general form of $(L')^{-1}$ for the path graph on n nodes is

$$(L')^{-1} = \frac{1}{E_{max}} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n-1 \end{pmatrix}_{(n-1)\times(n-1)} \tag{24}$$

$E_{\max} = 1$ for every path graph. Every sandpile on a path graph is indivisible as long as at least one vertex is unstable.

4.4 Indivisible Sandpiles

Definition 4.3. A sandpile on any graph is **indivisible** if and only if $u_d = u$.

4.4.1 Wheel Graph on $n = 2^k + 1$ vertices

The wheel graph on $n = 2^k + 1$ vertices is a special family of the wheel graphs. First, let's look at the maximum elementary divisors, in reference to 23.

	E_{max} prime factorization
$k = 2$	5, 3
$k = 3$	5, 3, 7
$k = 4$	5, 3, 7, 47
$k = 5$	5, 3, 7, 47, 2207
$k = 6$	5, 3, 7, 47, 1087, 2207, 4481
$\vdots$	$\vdots$

Conjecture 1. Let L_i be the i^{th} Lucas number. The maximum elementary divisor for a Wheel graph on $n = 2^k + 1$ vertices while $k \geq 2$ is

$$E_{max} = 5 \cdot \prod_{j=2}^{k} L_{2^{j-1}} \tag{25}$$

Conjecture 2. Let L_i be the i^{th} Lucas number. Then every $L_{2^{j-1}}$ is relatively prime, for $j \geq 1$.

Conjecture 3. Let $A = (L')^{-1} \cdot E_{\max}$ be a $2^k \times 2^k$ matrix. A sandpile σ on Γ will be indivisible if and only if the vertices $v \in V'$ satisfy conditions on the prime factorization of $E_{\max}$ for the given k.

Conjecture 4. For $i, \ell \in \mathbb{Z}$, set

$$\zeta(i, \ell) := 2^{\ell-1}(2i + 1) - 1$$

For $1 \leq \ell \leq k - 1$ and $0 \leq j \leq 2^{\ell} - 1$,

$$0 \mod L_{2^\ell} = \sum_{i=0}^{2^{k-l}-1} (-1)^i \left[L_0 x_{\zeta(i,\ell)+j} + \sum_{n=1}^{\ell-1} L_{2n}(x_{\zeta(i,\ell)+j+n} + x_{\zeta(i,\ell)+j-n} \right] \tag{26}$$

The four conjectures given above (1, 2, 3, and 4) are not proven.

4.4.2 Complete Graph

Theorem 5. *A sandpile $\sigma : V' \to \mathbb{Z}$ which satisfies $\sigma \geq d - 1$ on a complete graph K_n on $n \geq 3$ vertices is indivisible if and only if the difference between any two vertices $v_i, v_j \in V'$ is given by*

$$\sigma_i - \sigma_j = 0 \ mod \ n$$

Proof.

($\Leftarrow$) Assume $\sigma_i - \sigma_j = 0$ mod n for all i, j. Then there exists k such that, for all i, $\sigma_i = k$ mod n. From

15, and $(L')^{-1}$ from 20,

$$u = \frac{1}{n}\begin{pmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{pmatrix} \cdot \begin{pmatrix} \sigma_1 - n + 2 \\ \sigma_2 - n + 2 \\ \vdots \\ \sigma_{n-1} - n + 2 \end{pmatrix}$$

$$u_i = \frac{2(\sigma_i - n + 2)}{n} + \frac{\left(\sum_{x=1, x\neq i}^{n-1} (\sigma_x - n + 2)\right)}{n}$$

$$u_i = \frac{(\sigma_i - n + 2)}{n} + \frac{\left(\sum_{x=1}^{n-1} (\sigma_x - n + 2)\right)}{n}$$

$$u_i = \frac{\sigma_i + \left(\sum_{x=1}^{n-1} \sigma_x\right)}{n} - \frac{(n-2)(1 + (n-1))}{n}$$

$$u_i = \frac{\sigma_i + \left(\sum_{x=1}^{n-1} \sigma_x\right)}{n} - \frac{n(n-2)}{n}$$

$$u_i = \frac{\sigma_i + \left(\sum_{x=1}^{n-1} \sigma_x\right)}{n} - (n-2) \tag{27}$$

Let us consider just $\sigma_i + \left(\sum_{x=1}^{n-1} \sigma_x\right)$. We stated in the begining of the proof that there exists k such that, for all i, $\sigma_i = k \bmod n$.

$$\begin{aligned} \sigma_i + \left(\sum_{x=1}^{n-1} \sigma_x\right) &= k + k(n-1) \bmod n \\ &= k(1 + n - 1) \bmod n \\ &= k * n \bmod n \end{aligned}$$

If we consider the whole $\frac{\sigma_i + \left(\sum_{x=1}^{n-1} \sigma_x\right)}{n}$, then we have

$$\begin{aligned} \frac{\sigma_i + \left(\sum_{x=1}^{n-1} \sigma_x\right)}{n} &= \frac{k * n}{n} \bmod n \\ &= k \bmod n \end{aligned}$$

Now, applying modular arithmetic with 27, we have

$$u_i = k - n + 2 \bmod n$$

$$u_i = k + 2 \bmod n$$

We know $k \in \mathbb{Z}$, and hence $u \in \mathbb{Z}$.

($\Rightarrow$) Assume $u \in \mathbb{Z}$. From 15,

$$\sigma = L'u + d - 1 \text{ on a graph } \Gamma$$

$$\sigma = L'u + n - 2 \text{ on a complete graph } K_n$$

$$\sigma = \begin{pmatrix} n-1 & -1 & \dots & -1 \\ -1 & n-1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \dots & n-1 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \end{pmatrix} + \begin{pmatrix} n-2 \\ n-2 \\ \vdots \\ n-2 \end{pmatrix}$$

$$\sigma_i = (n-1)u_i - \left(\sum_{x=1, x\neq i}^{n-1} u_x \right) + n - 2$$

$$\sigma_i = n(u_i) + n - u_i - \left(\sum_{x=1, x\neq i}^{n-1} u_x \right) - 2$$

$$\sigma_i = n(u_i + 1) - 2 - \left(\sum_{x=1}^{n-1} u_x \right)$$

$$\sigma_j = n(u_j + 1) - 2 - \left(\sum_{x=1}^{n-1} u_x \right)$$

$$\sigma_i - \sigma_j = n(u_i + 1) - n(u_j + 1)$$

$$\sigma_i - \sigma_j = n(u_i - u_j)$$

$$\sigma_i - \sigma_j = n(u_i - u_j) \bmod n$$

We began with the assumption that all $u \in \mathbb{Z}$. Therefore,

$$\sigma_i - \sigma_j = 0 \bmod n$$

□

4.4.3 Wheel Graph

Theorem 6. *Let W_5 be a wheel graph on 5 vertices, with the sink located at the vertex with the highest degree. Label $v \in V'$ as a, b, c, d such that a, c are not adjacent and b, d are not adjacent. A sandpile $\sigma : V' \to \mathbb{R}$ on*

W_5 which satisfies $\sigma \geq 2$ is indivisible if and only if

$$a = c \ mod\ 3 \tag{28}$$

$$b = d \ mod\ 3 \tag{29}$$

$$a \neq b \ mod\ 3$$

$$a + c = b + d \ mod\ 5 \tag{30}$$

Proof.

($\Rightarrow$) Assume 28, 29, and 30. Let us consider some $\tilde{u}$. Because $\sigma \geq 2$, we know $\tilde{u} \geq 0$ and satisfies $\sigma - L'\tilde{u} = d - 1$. Then $\tilde{u} = u_d$ from Theorem 4.

Working with 15, we have

$$\tilde{u} = (L')^{-1}(\sigma - d + 1)$$

We know $(L')^{-1}$ from 23, and that $d - 1$ on any wheel graph is 2.

$$\tilde{u} = \frac{1}{15}\begin{pmatrix} 7 & 3 & 2 & 3 \\ 3 & 7 & 3 & 2 \\ 2 & 3 & 7 & 3 \\ 3 & 2 & 3 & 7 \end{pmatrix}(\sigma - 2)$$

Let $a = \sigma_1, b = \sigma_2, c = \sigma_3,$ and $d = \sigma_4$.

Consider the i^{th} component of $\tilde{u}$.

$$15\tilde{u}_i = 7\sigma_i + 3\sigma_{i+1} + 2\sigma_{i+2} + 3\sigma_{i+3} - 30$$

$$15\tilde{u}_i = \sigma_i + 2\sigma_{i+2} \text{ mod } 3$$

From 28 and 29, we can say that $\sigma_i = \sigma_{i+2}$ mod 3

$$15\tilde{u}_i = \sigma_i + 2\sigma_i \text{ mod } 3$$

$$15\tilde{u}_i = 3\sigma_i \text{ mod } 3$$

$$15\tilde{u}_i = 0 \text{ mod } 3 \Rightarrow \tilde{u}_i \in \mathbb{Z}$$

We have shown that $\tilde{u} \in \mathbb{Z}$ such that $\tilde{u} = u$. Then $\tilde{u} = u = u_d$ and therefore $\tilde{u}$ is the odometer and σ is indivisible.

($\Leftarrow$) Assume σ is indivisible such that $u = u_d$. Since $\sigma \geq 2$, then $\tilde{u} \geq 0$ and satisfies $\sigma - L' = d - 1$. From Theorem 4, $\tilde{u} = u_d = u$ is the odometer.

$$15\tilde{u}_i = 7\sigma_i + 3\sigma_{i+1} + 2\sigma_{i+2} + 3\sigma_{i+3} - 30$$

Let $7\sigma_i + 3\sigma_{i+1} + 2\sigma_{i+2} + 3\sigma_{i+3} - 30 = x$. For what conditions on x does $15|x$? The prime factorization of 15 is 3, 5. If $3|x$ and $5|x$ then $15|x$.

– Determine when $3|x$. There is some $i \in \mathbb{Z}$ such that $x = 3i$.

$$7\sigma_i + 3\sigma_{i+1} + 2\sigma_{i+2} + 3\sigma_{i+3} - 30 = 3i$$

$$7\sigma_i + 2\sigma_{i+2} = 3i$$

If $\sigma_{i+2} = \sigma_i + 3m$ for some $m \in \mathbb{Z}$, then

$$7\sigma_i + 2(\sigma_i + 3m) = 3i$$

$$9\sigma_i + 6m = 3i$$

$$6m = 3i$$

$$2m = i$$

$$3|x$$

– Determine when $5|x$. There is some $j \in \mathbb{Z}$ such that $x = 5j$.

$$7\sigma_i + 3\sigma_{i+1} + 2\sigma_{i+2} + 3\sigma_{i+3} - 30 = 5j$$

$$5\sigma_i + 2\sigma_i + \sigma_{i+1} + 2\sigma_{i+1} + 2\sigma_{i+2} + \sigma_{i+3} + 2\sigma_{i+3} - 30 = 5j$$

$$5\sigma_i + \sigma_{i+1} + \sigma_{i+3} + 2(\sigma_i + \sigma_{i+1} + \sigma_{i+2} + \sigma_{i+3}) - 30 = 5j$$

If $\sigma_i + \sigma_{i+2} = \sigma_{i+1} + \sigma_{i+3} + 5n$ for some $n \in \mathbb{Z}$, (equivalently, $\sigma_i + \sigma_{i+2} = \sigma_{i+1} + \sigma_{i+3} \bmod 5$, then

$$5\sigma_i + \sigma_{i+1} + \sigma_{i+3} + 2(2(\sigma_{i+1} + \sigma_{i+3}) + 5n) - 30 = 5j$$

$$5\sigma_i + 5\sigma_{i+1} + 5\sigma_{i+3} + 10n - 30 = 5j$$

$$\sigma_i + \sigma_{i+1} + \sigma_{i+3} + 2n - 6 = j$$

The left side is all integer valued, and the right side is an integer.

$$5|x$$

We have shown that, when $\sigma_i = \sigma_{i+2} \bmod 3$ and $\sigma_i + \sigma_{i+2} = \sigma_{i+1} + \sigma_{i+3} \bmod 5$, $3|x$ and $5|x$. Then $15|x$, satisfying that $\tilde{u} = u$.

□

The wheel graph on 6 vertices, W_6, has indivisible sandpiles which satisfy different conditions. One case is where all σ are equal mod 11, such that

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \sigma_5 \bmod 11$$

Another case is discussed in the following theorem.

Theorem 7. *The following conditions satisfy that σ be indivisible on W_6.*

$$\sigma_1 = \sigma_4 \ mod\ 11 \tag{31}$$

$$\sigma_2 = \sigma_3 \ mod\ 11 \tag{32}$$

$$\sigma_1 \neq \sigma_2 \ mod\ 11$$

$$2\sigma_1 + \sigma_2 = 3\sigma_5 \ mod\ 11 \tag{33}$$

$$3\sigma_2 + 2\sigma_5 = 5\sigma_1 \ mod\ 11 \tag{34}$$

Proof. Assume the four equations given above. Let $\sigma \geq 2$ such that $\tilde{u} \geq 0$ for $\tilde{u} : V' \to \mathbb{R}$. Then we know $\tilde{u}$

satisfies $\tilde{u} = (L')^{-1}(\sigma - d + 1)$. From 22, we have

$$(L')^{-1} = \frac{1}{11}\begin{pmatrix} 5 & 2 & 1 & 1 & 2 \\ 2 & 5 & 2 & 1 & 1 \\ 1 & 2 & 5 & 2 & 1 \\ 1 & 2 & 5 & 2 & 1 \\ 2 & 1 & 1 & 2 & 5 \end{pmatrix}$$

$$11\tilde{u} = \begin{pmatrix} 5 & 2 & 1 & 1 & 2 \\ 2 & 5 & 2 & 1 & 1 \\ 1 & 2 & 5 & 2 & 1 \\ 1 & 2 & 5 & 2 & 1 \\ 2 & 1 & 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} \sigma_1 - 2 \\ \sigma_2 - 2 \\ \sigma_3 - 2 \\ \sigma_4 - 2 \\ \sigma_5 - 2 \end{pmatrix}$$

We will prove that $\tilde{u} = u = u_d$ with one component of $\tilde{u}$ at a time.

$\boldsymbol{\tilde{u}_1}$:

$$11\tilde{u}_1 = 5\sigma_1 + 2\sigma_2 + \sigma_3 + \sigma_4 + 2\sigma_5$$

Apply 31 and 32

$$11\tilde{u}_1 = 5\sigma_1 + 2\sigma_2 + \sigma_2 + \sigma_1 + 2\sigma_5 \text{ mod } 11$$

$$11\tilde{u}_1 = 6\sigma_1 + 3\sigma_2 + 2\sigma_5 \text{ mod } 11$$

$$11\tilde{u}_1 = 3(2\sigma_1 + \sigma_2) + 2\sigma_5 \text{ mod } 11$$

Apply 33

$$11\tilde{u}_1 = 3(3\sigma_5) + 2\sigma_5 \text{ mod } 11$$

$$11\tilde{u}_1 = 11\sigma_5 \text{ mod } 11$$

$$11\tilde{u}_1 = 0 \text{ mod } 11$$

Then $\tilde{u}_1$ must be integer valued.

$\boldsymbol{\tilde{u}_2}$:

$$11\tilde{u}_2 = 2\sigma_1 + 5\sigma_2 + 2\sigma_3 + \sigma_4 + \sigma_5$$

Apply 31 and 32

$$11\tilde{u}_2 = 2\sigma_1 + 5\sigma_2 + 2\sigma_2 + \sigma_1 + \sigma_5 \text{ mod } 11$$

$$11\tilde{u}_2 = 3\sigma_1 + 7\sigma_2 + \sigma_5 \text{ mod } 11$$

$$11\tilde{u}_2 = \sigma_1 + (2\sigma_1 + \sigma_2) + 6\sigma_2 + \sigma_5 \text{ mod } 11$$

Apply 33

$$11\tilde{u}_2 = \sigma_1 + 6\sigma_2 + 4\sigma_5 \text{ mod } 11$$

$$11\tilde{u}_2 = \sigma_1 + 2(3\sigma_2 + 2\sigma_5) \text{ mod } 11$$

Apply 34

$$11\tilde{u}_2 = \sigma_1 + 2(5\sigma_1) \text{ mod } 11$$

$$11\tilde{u}_2 = 11\sigma_1 \text{ mod } 11$$

$$11\tilde{u}_2 = 0 \text{ mod } 11$$

Then $\tilde{u}_2$ must be integer valued.

The same method works for proving $\tilde{u}_3$, $\tilde{u}_4$, and $\tilde{u}_5$.

$\boldsymbol{\tilde{u}}$: All components of $\tilde{u}$ are integer valued. Then $\tilde{u}$ is integral, and therefore $\tilde{u} = u = u_d$ is the odometer.

□

4.4.4 Path Graph

Theorem 8. *All sandpiles $\sigma : V' \to \mathbb{Z}$ satisfying $\sigma \geq d - 1$ are indivisible.*

Proof. Let $\tilde{u} = (L')^{-1}(\sigma - d + 1)$.

$$\tilde{u} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2 & \dots & 2 \\ 1 & 2 & 3 & \dots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \dots & n-1 \end{pmatrix} (\sigma - d + 1)$$

From 24, the inverse reduced Laplacian for any path graph is integral. We know that $\sigma - d + 1$ is integral. Therefore, $\tilde{u} = u$ for all $\sigma \geq d - 1$. □

5 Sources in Graphs

We will define a source to be the only vertex in sandpile that chips are allowed to be placed on before stabilization. For example, before stabilization, a sandpile configuration with a source will have zero chips on every (non-sink) vertex, and any non-negative number of chips on the source.

5.1 Sources in Wheel Graphs

Conjecture: For wheel graphs, when the sink is the central vertex and the source is one of the vertices on the outside of the wheel, then the number of stable configurations reachable through stabilization is equal to the last non-zero entry in the diagonal of the corresponding laplacian matrix (i.e. the largest elementary divisor).

This was tested up to 30 vertices (after that the program takes a while to give an output), and in those cases the conjecture held.

Thus, we believe that if x is the last non-zero entry on the diagonal of a wheel graph's corresponding laplacian matrix, then, by placing chips on the source, there are are only x reachable stable configurations. In fact, if you repeatedly increment the number of the chips on the source by one, you will cycle through all of the reachable stable configurations.

5.2 'Super' Sources

We define a super source to be a source vertex that is connected to every non-sink vertex in a graph.

We define a super sink to be a sink vertex that is connected to every non-source vertex in a graph.

Conjecture: If a graph contains a super source and a super sink, then the number of stable sandpile configurations that can be reached by placing chips on the source is equal to the degree of the source (i.e. it is equal to the number of non-sink and non-sink vertices in the graph).

This was not proven, but it held for the examples it was tested on.

Since the sink and source are connect to all the same vertices, we can collapse them. Thus, this conjecture is about what happens when you allow a super sink to be the source for a sandpile.

5.3 Multiple Sources and Sinks

Nothing interesting was observed when multiple source vertices and/or multiple sink vertices are designated in a graph.

6 Induced Subgraphs

We wanted to explore the critical groups of induced subgraphs of strongly regular graphs. We looked at two questions:

Question 1: How many different critical groups are there for the v different subgraphs formed by taking a strongly regular graph Γ and removing one vertex along with all of its neighbors from Γ.
To do this, we wrote a program that would start with vertex 0, remove that vertex and its neighbors from Γ, and then note the corresponding critical group. Then the original Γ is restored and the process starts again with vertex 1. This repeats for every vertex. We kept track of which induced subgraphs produced the same critical groups as other induced subgraphs.

The results of doing this are summarized in the following table:

(v, k ,λ, μ)	Number of critical groups for subgraphs
(9, 4, 1, 2)	1
(10, 3, 0, 1)	1
(10, 6, 3, 4)	1
(13, 6, 2, 3)	1
(15, 6, 1, 3)	1
(15, 8, 4, 4)	1
(16, 5, 0, 2)	1
(16, 10, 6, 6)	1
(16, 6, 2, 2)	1
(16, 9, 4, 6)	1
(17, 8, 3, 4)	1
(21, 10, 3, 6)	1
(21, 10, 5, 4)	1
(25, 8, 3, 2)	1
(25, 16, 9, 12)	1
(25, 12, 5, 6)	1
(26, 10, 3, 4)	2
(26, 15, 8, 9)	2
(27, 10, 1, 5)	1
(27, 16, 10, 8)	1
(28, 12, 6, 4)	1
(28, 15, 6, 10)	1
(29, 14, 6, 7)	1
(35, 16, 6, 8)	2
(35, 18, 9, 9)	2
(36, 10, 4, 2)	1
(36, 25, 16, 20)	1
(36, 14, 4, 6)	12
(36, 21, 12, 12)	10
(36, 14, 7, 4)	1
(36, 21, 10, 15)	1
(36, 15, 6, 6)	1
(36, 20, 10, 12)	1
(37, 18, 8, 9)	1
(40, 12, 2, 4)	1
(40, 27, 18, 18)	1
(41, 20, 9, 10)	1
(45, 12, 3, 3)	1
(45, 16, 8, 4)	1
(45, 28, 15, 21)	1
(45, 22, 10, 11)	5
(49, 12, 5, 2)	1
(49, 36, 25, 30)	1
(49, 18, 7, 6)	1
(49, 30, 17, 20)	1
(49, 24, 11, 12)	1
(50, 7, 0, 1)	1
(50, 42, 35, 36)	1
(50, 21, 8, 9)	2
(50, 28, 15, 16)	2
(53, 26, 12, 13)	1
(55, 18, 9, 4)	1
(55, 36, 21, 28)	1
(56, 10, 0, 2)	1
(56, 45, 36, 36)	1
(57, 24, 11, 9)	21
(57, 32, 16, 20)	21
(61, 30, 14, 15)	1
(63, 30, 13, 15)	1

(63, 32, 16, 16)	1
(64, 14, 6, 2)	1
(64, 49, 36, 42)	1
(64, 18, 2, 6)	1
(64, 45, 32, 30)	1
(64, 21, 8, 6)	1
(64, 42, 26, 30)	1
(64, 27, 10, 12)	1
(64, 36, 20, 20)	1
(64, 28, 12, 12)	1
(64, 35, 18, 20)	1
(66, 20, 10, 4)	1
(66, 45, 28, 36)	1
(70, 27, 12, 9)	2
(70, 42, 23, 28)	2
(73, 36, 17, 18)	1
(77, 16, 0, 4)	1
(77, 60, 47, 45)	1
(78, 22, 11, 4)	1
(78, 55, 36, 45)	1
(81, 16, 7, 2)	1
(81, 64, 49, 56)	1
(81, 20, 1, 6)	1
(81, 60, 45, 42)	1
(81, 24, 9, 6)	1
(81, 56, 37, 42)	1
(81, 30, 9, 12)	1
(81, 50, 31, 30)	1
(81, 32, 13, 12)	1
(81, 48, 27, 30)	1

(81, 40, 19, 20)	1
(82, 36, 15, 16)	2
(82, 45, 24, 25)	2
(85, 20, 3, 5)	1
(85, 64, 48, 48)	1
(89, 44, 21, 22)	1
(91, 24, 12, 4)	1
(91, 66, 45, 55)	1
(96, 19, 2, 4)	1
(96, 76, 60, 60)	1
(96, 20, 4, 4)	1
(96, 75, 58, 60)	1
(97, 48, 23, 24)	1
(99, 48, 22, 24)	2
(99, 50, 25, 25)	2
(100, 18, 8, 2)	1
(100, 81, 64, 72)	1
(100, 22, 0, 6)	1
(100, 77, 60, 56)	1
(100, 27, 10, 6)	1
(100, 72, 50, 56)	1
(100, 33, 14, 9)	36
(100, 66, 41, 48)	36
(100, 36, 14, 12)	19
(100, 63, 38, 42)	19

Question 2: How many different critical groups are there for the v different subgraphs formed by taking a strongly regular graph Γ and removing a vertex from Γ?
The program to do this was similar to the one previously mentioned, except only a single vertex was deleted in each iteration, instead of a vertex and all of its neighbors.

The resulting number of different critical groups for the subgraphs of strongly regular graphs was the same as the numbers shown in the previous table, except in one case. For srg(36, 14, 4, 6), there were actually 13 different critical groups instead of only 12. We are unsure as to why this is the case and if it has any significance.

7 Appendix

Existence and primeness of Laplacian Eigenvalues of Strongly Regular Graphs

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
5	2	0	1	1.38196601125011	3.61803398874989	Exists	Error
9	4	1	2	3	6	Exists	Square Free
10	3	0	1	2	5	Exists	Relatively prime
	6	3	4	5	8	Exists	Relatively prime
13	6	2	3	4.69722436226801	8.30277563773199	Exists	Error
15	6	1	3	5	9	Exists	Relatively prime
	8	4	4	6	10	Exists	Square Free
16	5	0	2	4	8	Exists	NA
	10	6	6	8	12	Exists	NA
16	6	2	2	4	8	Exists	NA
	9	4	6	8	12	Exists	NA
17	8	3	4	6.43844718719117	10.5615528128088	Exists	Error
21	10	3	6	9	14	Exists	Relatively prime
	10	5	4	7	12	Exists	Relatively prime
21	10	4	5	8.20871215252208	12.7912878474779	DNE	Error
25	8	3	2	5	10	Exists	Square Free
	16	9	12	15	20	Exists	Square Free
25	12	5	6	10	15	Exists	Square Free
26	10	3	4	8	13	Exists	Relatively prime
	15	8	9	13	18	Exists	Relatively prime
27	10	1	5	9	15	Exists	P, P2
	16	10	8	12	18	Exists	P, P2
28	9	0	4	8	14	DNE	P, P2
	18	12	10	14	20	DNE	P, P2
28	12	6	4	8	14	Exists	P, P2
	15	6	10	14	20	Exists	P, P2
29	14	6	7	11.8074175964327	17.1925824035673	Exists	Error
33	16	7	8	13.627718676731	19.372281323269	DNE	Error

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
35	16	6	8	14	20	Exists	P, P2
	18	9	9	15	21	Exists	Square Free
36	10	4	2	6	12	Exists	P, P2
	25	16	20	24	30	Exists	P, P2
36	14	4	6	12	18	Exists	P, P2
	21	12	12	18	24	Exists	P, P2
36	14	7	4	9	16	Exists	Relatively prime
	21	10	15	20	27	Exists	Relatively prime
36	15	6	6	12	18	Exists	P, P2
	20	10	12	18	24	Exists	P, P2
37	18	8	9	15.4586187348509	21.5413812651491	Exists	Error
40	12	2	4	10	16	Exists	P, P2
	27	18	18	24	30	Exists	P, P2
41	20	9	10	17.2984378812836	23.7015621187164	Exists	Error
45	12	3	3	9	15	Exists	P, P2
	32	22	24	30	36	Exists	P, P2
45	16	8	4	10	18	Exists	Square Free
	28	15	21	27	35	Exists	Relatively prime
45	22	10	11	19.1458980337503	25.8541019662497	Exists	Error
49	12	5	2	7	14	Exists	Square Free
	36	25	30	35	42	Exists	Square Free
49	16	3	6	14	21	DNE	Square Free
	32	21	20	28	35	DNE	Square Free
49	18	7	6	14	21	Exists	Square Free
	30	17	20	28	35	Exists	Square Free
49	24	11	12	21	28	Exists	Square Free
50	7	0	1	5	10	Exists	Square Free
	42	35	36	40	45	Exists	Square Free
50	21	4	12	20	30	DNE	P, P2
	28	18	12	20	30	DNE	P, P2

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
50	21	8	9	18	25	Exists	Relatively prime
	28	15	16	25	32	Exists	Relatively prime
53	26	12	13	22.8599450553597	30.1400549446403	Exists	Error
55	18	9	4	11	20	Exists	Relatively prime
	36	21	28	35	44	Exists	Relatively prime
56	10	0	2	8	14	Exists	P, P2
	45	36	36	42	48	Exists	P, P2
56	22	3	12	21	32	DNE	Relatively prime
	33	22	15	24	35	DNE	Relatively prime
57	14	1	4	12	19	DNE	Relatively prime
	42	31	30	38	45	DNE	Relatively prime
57	24	11	9	19	27	Exists	Relatively prime
	32	16	20	30	38	Exists	Square Free
57	28	13	14	24.7250827823646	32.2749172176354	DNE	Error
61	30	14	15	26.5948751620467	34.4051248379533	Exists	Error
63	22	1	11	21	33	DNE	Square Free
	40	28	20	30	42	DNE	Square Free
63	30	13	15	27	35	Exists	Relatively prime
	32	16	16	28	36	Exists	NA
64	14	6	2	8	16	Exists	NA
	49	36	42	48	56	Exists	NA
64	18	2	6	16	24	Exists	NA
	45	32	30	40	48	Exists	NA
64	21	0	10	20	32	DNE	NA
	42	30	22	32	44	DNE	NA
64	21	8	6	16	24	Exists	NA
	42	26	30	40	48	Exists	NA
64	27	10	12	24	32	Exists	NA
	36	20	20	32	40	Exists	NA
64	28	12	12	24	32	Exists	NA
	35	18	20	32	40	Exists	NA

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
64	30	18	10	20	32	DNE	NA
	33	12	22	32	44	DNE	NA
65	32	15	16	28.4688711258507	36.5311288741493	Unknown	Error
66	20	10	4	12	22	Exists	P, P2
	45	28	36	44	54	Exists	P, P2
69	20	7	5	15	23	Unknown	Relatively prime
	48	32	36	46	54	Unknown	Square Free
69	34	16	17	30.346688068541	38.653311931459	DNE	Error
70	27	12	9	21	30	Exists	Square Free
	42	23	28	40	49	Exists	Relatively prime
73	36	17	18	32.2279981273412	40.7720018726588	Exists	Error
75	32	10	16	30	40	DNE	P, P2
	42	25	21	35	45	DNE	Square Free
76	21	2	7	19	28	DNE	Relatively prime
	54	39	36	48	57	DNE	Square Free
76	30	8	14	28	38	DNE	P, P2
	45	28	24	38	48	DNE	P, P2
76	35	18	14	28	38	DNE	P, P2
	40	18	24	38	48	DNE	P, P2
77	16	0	4	14	22	Exists	Square Free
	60	47	45	55	63	Exists	Relatively prime
77	38	18	19	34.1125178063039	42.8874821936961	DNE	Error
78	22	11	4	13	24	Exists	Relatively prime
	55	36	45	54	65	Exists	Relatively prime
81	16	7	2	9	18	Exists	NA
	64	49	56	63	72	Exists	NA
81	20	1	6	18	27	Exists	NA
	60	45	42	54	63	Exists	NA
81	24	9	6	18	27	Exists	NA
	56	37	42	54	63	Exists	NA

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
81	30	9	12	27	36	Exists	NA
	50	31	30	45	54	Exists	NA
81	32	13	12	27	36	Exists	NA
	48	27	30	45	54	Exists	NA
81	40	13	26	39	54	DNE	P, P2
	40	25	14	27	42	DNE	P, P2
81	40	19	20	36	45	Exists	NA
82	36	15	16	32	41	Exists	Relatively prime
	45	24	25	41	50	Exists	Relatively prime
85	14	3	2	10	17	Unknown	Relatively prime
	70	57	60	68	75	Unknown	Relatively prime
85	20	3	5	17	25	Exists	Relatively prime
	64	48	48	60	68	Exists	NA
85	30	11	10	25	34	Unknown	Relatively prime
	54	33	36	51	60	Unknown	Square Free
85	42	20	21	37.8902277713536	47.1097722286464	Unknown	Error
88	27	6	9	24	33	Unknown	Square Free
	60	41	40	55	64	Unknown	Relatively prime
89	44	21	22	39.7830094339717	49.2169905660283	Exists	Error
91	24	12	4	14	26	Exists	Square Free
	66	45	55	65	77	Exists	Relatively prime
93	46	22	23	41.6781746195035	51.3218253804965	DNE	Error
95	40	12	20	38	50	DNE	Square Free
	54	33	27	45	57	DNE	P, P2
96	19	2	4	16	24	Exists	NA
	76	60	60	72	80	Exists	NA
96	20	4	4	16	24	Exists	NA
	75	58	60	72	80	Exists	NA
96	35	10	14	32	42	Unknown	P, P2
	60	38	36	54	64	Unknown	P, P2
96	38	10	18	36	48	DNE	NA
	57	36	30	48	60	DNE	NA

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
96	45	24	18	36	48	DNE	NA
	50	22	30	48	60	DNE	NA
97	48	23	24	43.5755710991019	53.424428900898	Exists	Error
99	14	1	2	11	18	Unknown	Relatively prime
	84	71	72	81	88	Unknown	Relatively prime
99	42	21	15	33	45	Unknown	P, P2
	56	28	36	54	66	Unknown	P, P2
99	48	22	24	44	54	Exists	P, P2
	50	25	25	45	55	Exists	Square Free
100	18	8	2	10	20	Exists	P, P2
	81	64	72	80	90	Exists	P, P2
100	22	0	6	20	30	Exists	P, P2
	77	60	56	70	80	Exists	P, P2
100	27	10	6	20	30	Exists	P, P2
	72	50	56	70	80	Exists	P, P2
100	33	8	12	30	40	Unknown	P, P2
	66	44	42	60	70	Unknown	P, P2
100	33	14	9	25	36	Exists	Relatively prime
	66	41	48	64	75	Exists	Relatively prime
100	33	18	7	20	35	DNE	Square Free
	66	39	52	65	80	DNE	Square Free
100	36	14	12	30	40	Exists	P, P2
	63	38	42	60	70	Exists	P, P2
100	44	18	20	40	50	Exists	P, P2
	55	30	30	50	60	Exists	P, P2
100	45	20	20	40	50	Exists	P, P2
	54	28	30	50	60	Exists	P, P2
101	50	24	25	45.4750621894396	55.5249378105605	Exists	Error
105	26	13	4	15	28	Exists	Relatively prime
	78	55	66	77	90	Exists	Relatively prime
105	32	4	12	30	42	Exists	Square Free
	72	51	45	63	75	Exists	P, P2

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
105	40	15	15	35	45	Unknown	Square Free
	64	38	40	60	70	Unknown	P, P2
105	52	21	30	50	63	Unknown	Relatively prime
	52	29	22	42	55	Unknown	Relatively prime
105	52	25	26	47.3765246170202	57.6234753829798	DNE	Error
109	54	26	27	49.2798467455447	59.7201532544553	Exists	Error
111	30	5	9	27	37	Unknown	Relatively prime
	80	58	56	74	84	Unknown	P, P2
111	44	19	16	37	48	Exists	Relatively prime
	66	37	42	63	74	Exists	Relatively prime
112	30	2	10	28	40	Exists	NA
	81	60	54	72	84	Exists	NA
112	36	10	12	32	42	Unknown	P, P2
	75	50	50	70	80	Unknown	P, P2
113	56	27	28	51.1849270936327	61.8150729063673	Exists	Error
115	18	1	3	15	23	Unknown	Relatively prime
	96	80	80	92	100	Unknown	NA
117	36	15	9	27	39	Exists	P, P2
	80	52	60	78	90	Exists	P, P2
117	58	28	29	53.091673086804	63.908326913196	Unknown	Error
119	54	21	27	51	63	Exists	P, P2
	64	36	32	56	68	Exists	NA
120	28	14	4	16	30	Exists	P, P2
	91	66	78	90	104	Exists	P, P2
120	34	8	10	30	40	Unknown	P, P2
	85	60	60	80	90	Unknown	P, P2
120	35	10	10	30	40	Unknown	P, P2
	84	58	60	80	90	Unknown	P, P2
120	42	8	18	40	54	Exists	P, P2
	77	52	44	66	80	Exists	P, P2
120	51	18	24	48	60	Exists	NA
	68	40	36	60	72	Exists	NA

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
120	56	28	24	48	60	Exists	NA
	63	30	36	60	72	Exists	NA
121	20	9	2	11	22	Exists	Square Free
	100	81	90	99	110	Exists	Square Free
121	30	11	6	22	33	Exists	Square Free
	90	65	72	88	99	Exists	Square Free
121	36	7	12	33	44	Unknown	Square Free
	84	59	56	77	88	Unknown	Square Free
121	40	15	12	33	44	Exists	Square Free
	80	51	56	77	88	Exists	Square Free
121	48	17	20	44	55	Unknown	Square Free
	72	43	42	66	77	Unknown	Square Free
121	50	21	20	44	55	Exists	Square Free
	70	39	42	66	77	Exists	Square Free
121	56	15	35	55	77	DNE	Square Free
	64	42	24	44	66	DNE	P, P2
121	60	29	30	55	66	Exists	Square Free
122	55	24	25	50	61	Exists	Relatively prime
	66	35	36	61	72	Exists	Relatively prime
125	28	3	7	25	35	Exists	P, P2
	96	74	72	90	100	Exists	P, P2
125	48	28	12	30	50	DNE	P, P2
	76	39	57	75	95	DNE	P, P2
125	52	15	26	50	65	Exists	P, P2
	72	45	36	60	75	Exists	P, P2
125	62	30	31	56.9098300562505	68.0901699437495	Exists	Error
126	25	8	4	18	28	Exists	P, P2
	100	78	84	98	108	Exists	P, P2
126	45	12	18	42	54	Exists	P, P2
	80	52	48	72	84	Exists	NA
126	50	13	24	48	63	Exists	P, P2
	75	48	39	63	78	Exists	P, P2

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
126	60	33	24	48	63	Exists	P, P2
	65	28	39	63	78	Exists	P, P2
129	64	31	32	58.8210916541997	70.1789083458003	DNE	Error
130	48	20	16	40	52	Exists	NA
	81	48	54	78	90	Exists	P, P2
133	24	5	4	19	28	Unknown	Relatively prime
	108	87	90	105	114	Unknown	Square Free
133	32	6	8	28	38	Unknown	P, P2
	100	75	75	95	105	Unknown	Square Free
133	44	15	14	38	49	Unknown	Relatively prime
	88	57	60	84	95	Unknown	Relatively prime
133	66	32	33	60.7337187026646	72.2662812973354	DNE	Error
135	64	28	32	60	72	Exists	NA
	70	37	35	63	75	Exists	P, P2
136	30	8	6	24	34	Unknown	P, P2
	105	80	84	102	112	Unknown	P, P2
136	30	15	4	17	32	Exists	Relatively prime
	105	78	91	104	119	Exists	Relatively prime
136	60	24	28	56	68	Exists	NA
	75	42	40	68	80	Exists	NA
136	63	30	28	56	68	Exists	NA
	72	36	40	68	80	Exists	NA
137	68	33	34	62.6476500446402	74.3523499553598	Exists	Error
141	70	34	35	64.562828956481	76.437171043519	DNE	Error
143	70	33	35	65	77	Exists	Relatively prime
	72	36	36	66	78	Exists	Square Free
144	22	10	2	12	24	Exists	NA
	121	100	110	120	132	Exists	NA
144	33	12	6	24	36	Exists	NA
	110	82	90	108	120	Exists	NA
144	39	6	12	36	48	Exists	NA
	104	76	72	96	108	Exists	NA

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
144	44	16	12	36	48	Exists	NA
	99	66	72	96	108	Exists	NA
144	52	16	20	48	60	Unknown	NA
	91	58	56	84	96	Unknown	NA
144	55	22	20	48	60	Exists	NA
	88	52	56	84	96	Exists	NA
144	65	16	40	64	90	DNE	P, P2
	78	52	30	54	80	DNE	P, P2
144	65	28	30	60	72	Exists	NA
	78	42	42	72	84	Exists	NA
144	66	30	30	60	72	Exists	NA
	77	40	42	72	84	Exists	NA
145	72	35	36	66.4792027106039	78.5207972893961	Unknown	Error
147	66	25	33	63	77	Unknown	Square Free
	80	46	40	70	84	Unknown	P, P2
148	63	22	30	60	74	Unknown	P, P2
	84	50	44	74	88	Unknown	P, P2
148	70	36	30	60	74	Unknown	P, P2
	77	36	44	74	88	Unknown	P, P2
149	74	36	37	68.3967221921332	80.6032778078668	Exists	Error
153	32	16	4	18	34	Exists	Square Free
	120	91	105	119	135	Exists	Relatively prime
153	56	19	21	51	63	Unknown	P, P2
	96	60	60	90	102	Unknown	P, P2
153	76	37	38	70.3153415615735	82.6846584384265	Unknown	Error
154	48	12	16	44	56	Unknown	NA
	105	72	70	98	110	Unknown	Square Free
154	51	8	21	49	66	DNE	Relatively prime
	102	71	60	88	105	DNE	Relatively prime
154	72	26	40	70	88	Unknown	P, P2
	81	48	36	66	84	Unknown	P, P2

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
155	42	17	9	31	45	Exists	Relatively prime
	112	78	88	110	124	Exists	P, P2
156	30	4	6	26	36	Exists	P, P2
	125	100	100	120	130	Exists	P, P2
157	78	38	39	72.2350179569292	84.7649820430708	Exists	Error
160	54	18	18	48	60	Unknown	NA
	105	68	70	100	112	Unknown	NA
161	80	39	40	74.1557112297752	86.8442887702248	DNE	Error
162	21	0	3	18	27	Unknown	NA
	140	121	120	135	144	Unknown	NA
162	23	4	3	18	27	Unknown	NA
	138	117	120	135	144	Unknown	NA
162	49	16	14	42	54	Unknown	P, P2
	112	76	80	108	120	Unknown	NA
162	56	10	24	54	72	Exists	NA
	105	72	60	90	108	Exists	NA
162	69	36	24	54	72	Unknown	NA
	92	46	60	90	108	Unknown	NA
165	36	3	9	33	45	Exists	P, P2
	128	100	96	120	132	Exists	NA
165	82	40	41	76.0773837106674	88.9226162893326	DNE	Error
169	24	11	2	13	26	Exists	Error
	144	121	132	143	156	Exists	Error
169	36	13	6	26	39	Exists	Error
	132	101	110	130	143	Exists	Error
169	42	5	12	39	52	Unknown	Error
	126	95	90	117	130	Unknown	Error
169	48	17	12	39	52	Exists	Error
	120	83	90	117	130	Exists	Error
169	56	15	20	52	65	Unknown	Error
	112	75	72	104	117	Unknown	Error

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
169	60	23	20	52	65	Exists	Error
	108	67	72	104	117	Exists	Error
169	70	27	30	65	78	Unknown	Error
	98	57	56	91	104	Unknown	Error
169	72	31	30	65	78	Exists	Error
	96	53	56	91	104	Exists	Error
169	84	41	42	78	91	Exists	Error
170	78	35	36	72	85	Exists	Relatively prime
	91	48	49	85	98	Exists	Relatively prime
171	34	17	4	19	36	Exists	Relatively prime
	136	105	120	135	152	Exists	Relatively prime
171	50	13	15	45	57	Unknown	P, P2
	120	84	84	114	126	Unknown	P, P2
171	60	15	24	57	72	Unknown	P, P2
	110	73	66	99	114	Unknown	P, P2
173	86	42	43	79.9235267810171	93.076473218983	Exists	Error
175	30	5	5	25	35	Exists	P, P2
	144	118	120	140	150	Exists	P, P2
175	66	29	22	55	70	Unknown	Square Free
	108	63	72	105	120	Unknown	Square Free
175	72	20	36	70	90	Exists	Square Free
	102	65	51	85	105	Exists	Square Free
176	25	0	4	22	32	Unknown	P, P2
	150	128	126	144	154	Unknown	P, P2
176	40	12	8	32	44	Exists	NA
	135	102	108	132	144	Exists	NA
176	45	18	9	33	48	Exists	Square Free
	130	93	104	128	143	Exists	Relatively prime
176	49	12	14	44	56	Exists	NA
	126	90	90	120	132	Exists	NA
176	70	18	34	68	88	Exists	NA
	105	68	54	88	108	Exists	NA

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
176	70	24	30	66	80	Unknown	P, P2
	105	64	60	96	110	Unknown	P, P2
176	70	42	18	44	72	DNE	NA
	105	52	78	104	132	DNE	NA
176	85	48	34	68	88	Exists	NA
	90	38	54	88	108	Exists	NA
177	88	43	44	81.847932652175	95.152067347825	DNE	Error
181	90	44	45	83.7731879764631	97.2268120235369	Exists	Error
183	52	11	16	48	61	Unknown	Relatively prime
	130	93	90	122	135	Unknown	Relatively prime
183	70	29	25	61	75	Exists	Relatively prime
	112	66	72	108	122	Exists	P, P2
184	48	2	16	46	64	DNE	P, P2
	135	102	90	120	138	DNE	P, P2
185	92	45	46	85.6992647456323	99.3007352543677	Unknown	Error
189	48	12	12	42	54	Unknown	P, P2
	140	103	105	135	147	Unknown	P, P2
189	60	27	15	45	63	Unknown	NA
	128	82	96	126	144	Unknown	NA
189	88	37	44	84	99	Unknown	P, P2
	100	55	50	90	105	Unknown	P, P2
189	94	46	47	87.6261364575663	101.373863542434	DNE	Error
190	36	18	4	20	38	Exists	P, P2
	153	120	136	152	170	Exists	P, P2
190	45	12	10	38	50	Unknown	Square Free
	144	108	112	140	152	Unknown	NA
190	84	33	40	80	95	Unknown	Square Free
	105	60	55	95	110	Unknown	Square Free
190	84	38	36	76	90	Exists	P, P2
	105	56	60	100	114	Exists	P, P2
190	90	45	40	80	95	Unknown	Square Free
	99	48	55	95	110	Unknown	Square Free

v	k	λ	μ	Laplacian	Eigenvalues	Existence	Primeness
193	96	47	48	89.5537780052751	103.446221994725	Exists	Error
195	96	46	48	90	104	Exists	P, P2
	98	49	49	91	105	Exists	Square Free
196	26	12	2	14	28	Exists	P, P2
	169	144	156	168	182	Exists	P, P2
196	39	2	9	36	49	Unknown	Relatively prime
	156	125	120	147	160	Unknown	Relatively prime
196	39	14	6	28	42	Exists	P, P2
	156	122	132	154	168	Exists	P, P2
196	45	4	12	42	56	Unknown	P, P2
	150	116	110	140	154	Unknown	P, P2
196	52	18	12	42	56	Exists	P, P2
	143	102	110	140	154	Exists	P, P2
196	60	14	20	56	70	Exists	P, P2
	135	94	90	126	140	Exists	P, P2
196	60	23	16	49	64	Exists	Relatively prime
	135	90	99	132	147	Exists	Square Free
196	65	24	20	56	70	Exists	P, P2
	130	84	90	126	140	Exists	P, P2
196	75	26	30	70	84	Unknown	P, P2
	120	74	72	112	126	Unknown	P, P2
196	78	32	30	70	84	Exists	P, P2
	117	68	72	112	126	Exists	P, P2
196	81	42	27	63	84	Unknown	P, P2
	114	59	76	112	133	Unknown	Square Free
196	85	18	51	84	119	DNE	Square Free
	110	75	44	77	112	DNE	Square Free
196	90	40	42	84	98	Unknown	P, P2
	105	56	56	98	112	Unknown	P, P2
196	91	42	42	84	98	Exists	P, P2
	104	54	56	98	112	Exists	P, P2
197	98	48	49	91.4821655761909	105.517834423809	Exists	Error

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