A New Generalization of the Line Graph and its Spectral Characteristics

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James Madison University

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Preliminaries

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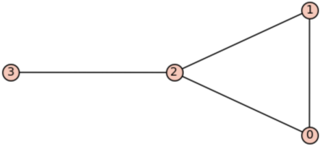

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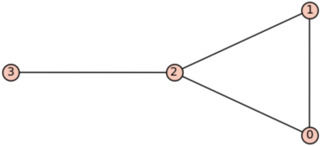

$$A = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

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The **spectrum** of a graph Γ, is the set of eigenvalues of the adjacency matrix of Γ along with their multiplicities. It is denoted $\lambda_1^{\alpha_1}, \ldots, \lambda_m^{\alpha_m}$ where the exponent represents the multiplicity of that eigenvalue.

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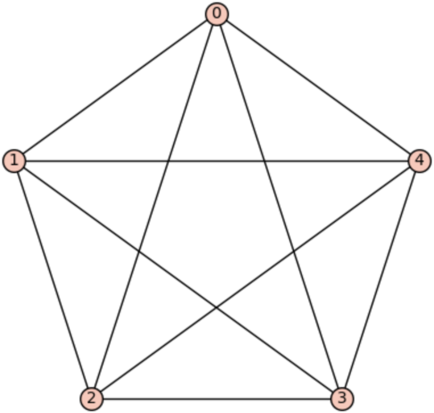
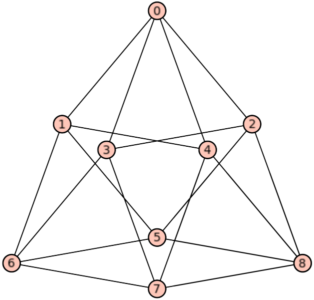


Motivating Example



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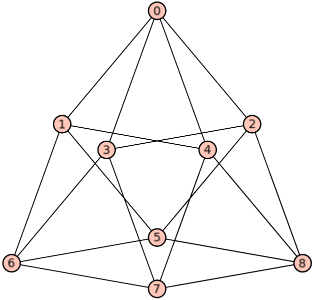

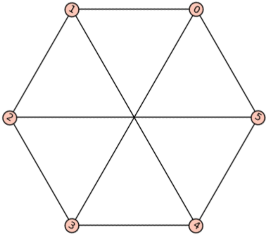

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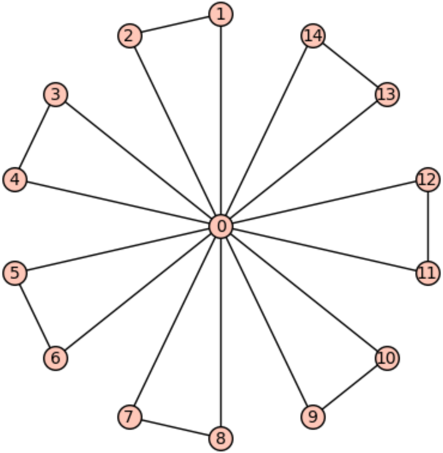

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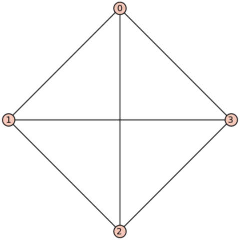

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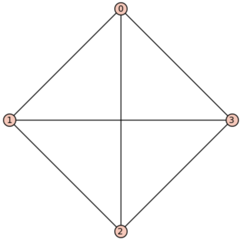

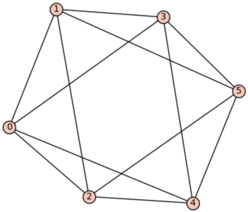

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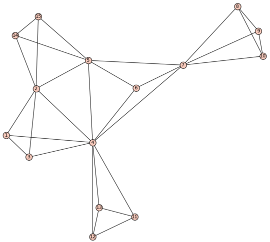

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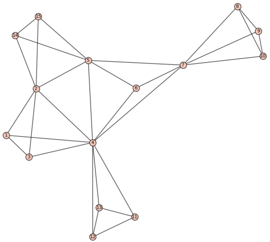

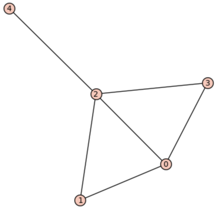

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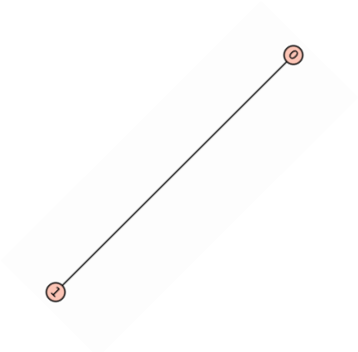

Since a 2-clique is a single edge.

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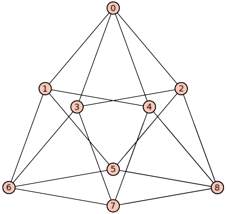

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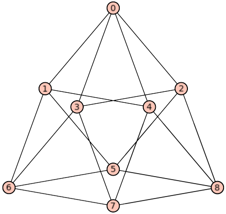

Lemma

If Γ is ω-clique regular, then $\omega = 2$ or ω is the order of Γ's largest clique.

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Spectral Theorems

Theorem

Suppose Γ is ω-clique regular and $L(\Gamma)$ has eigenvalues $\mu_1 \leq \cdots \leq \mu_m$. Then every eigenvalue λ of $C_\omega(\Gamma)$ is bounded by

$$\frac{\omega}{\omega - 1}\left(\frac{\mu_1}{2} + 1\right) - \omega \leq \lambda \leq \frac{\omega}{\omega - 1}\left(\frac{\mu_m}{2} + 1\right) - \omega.$$

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$\Delta(\Gamma)$ is the largest degree of a vertex in Γ.

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These bounds are the same and tight if Γ is regular.

Spectral Theorems

$p(\Gamma;\lambda)$ is the characteristic polynomial of the adjacency matrix of Γ.

Theorem

Suppose Γ is ω-clique regular and k-regular on n vertices. Then

$$p(C_\omega(\Gamma);\lambda) = (\lambda+\omega)^{\frac{nk}{\omega(\omega-1)}-n} p\left(\Gamma;\lambda+\omega-\frac{k}{\omega-1}\right).$$

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So if Γ has spectrum

$$\lambda_1^{\alpha_1},\ldots,\lambda_m^{\alpha_m},k^1$$

then $C_\omega(\Gamma)$ has spectrum

$$-\omega^{\frac{nk}{\omega(\omega-1)}-n},\left(\frac{k}{\omega-1}+\lambda_1-\omega\right)^{\alpha_1},\ldots,\ \omega\left(\frac{k}{\omega-1}-1\right)^1.$$

Applications to Conway's 99-graph

The spectrum of the 3-clique graph of Conway's 99-graph is

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Conway's 99-graph is an srg(99,14,1,2).

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Questions?

Acknowledgments

We are grateful to the Robert E. Tickle foundation for funding our research, and to Dr. Joshua Ducey for mentoring us.