Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

The quilted Atiyah-Floer conjecture and the Yang-Mills heat flow

David L. Duncan

Michigan State University

2015 AMS Spring Meeting - Georgetown University

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

Notation

Notation

Y, a closed, connected, oriented 3-manifold

Assume $b_1(Y) > 0$

Notation

Y, a closed, connected, oriented 3-manifold

Assume $b_1(Y) > 0$

$P \to Y$, an $\mathrm{SO}(3)$-bundle

Assume $w_2(P) \in H^2(Y, \mathbb{Z}_2)$ is in the image of a generator of $H^2(Y, \mathbb{Z})/\mathrm{torsion}$

The conjecture

The conjecture

$$HF_{\text{inst}}(Y)$$

defined by counting (isolated) instantons on $\mathbb{R} \times Y$

The conjecture

$HF_{\text{inst}}(Y)$

defined by counting (isolated) instantons on $\mathbb{R} \times Y$

$HF_{\text{symp}}(Y)$

defined by counting (isolated) holomorphic strips $\mathbb{R} \times I \to M$, in a certain symplectic manifold M

The conjecture

Quilted Atiyah-Floer conjecture

$$HF_{\text{inst}}(Y) \overset{?}{\cong} HF_{\text{symp}}(Y)$$

$HF_{\text{inst}}(Y)$	$HF_{\text{symp}}(Y)$
defined by counting (isolated) instantons on $\mathbb{R} \times Y$	defined by counting (isolated) holomorphic strips $\mathbb{R} \times I \to M$, in a certain symplectic manifold M

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

How a proof might go

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

How a proof might go

$\mathcal{M}_{\text{inst}}(g) :=$ moduli space of isolated instantons on $\mathbb{R} \times Y$

How a proof might go

$\mathcal{M}_{\text{inst}}(g) :=$ moduli space of isolated instantons on $\mathbb{R} \times Y$

Depends on a choice of metric g on Y, but only up to cobordism.

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

How a proof might go

$\mathcal{M}_{\text{inst}}(g) :=$ moduli space of isolated instantons on $\mathbb{R} \times Y$
Depends on a choice of metric g on Y, but only up to cobordism.

$\mathcal{M}_{\text{symp}}(g) :=$ moduli space of isolated holomorphic strips in M

How a proof might go

$\mathcal{M}_{\text{inst}}(g) :=$ moduli space of isolated instantons on $\mathbb{R} \times Y$
Depends on a choice of metric g on Y, but only up to cobordism.

$\mathcal{M}_{\text{symp}}(g) :=$ moduli space of isolated holomorphic strips in M
Depends on a choice of metric g, but only up to cobordism.

How a proof might go

$\mathcal{M}_{\text{inst}}(g) :=$ moduli space of isolated instantons on $\mathbb{R} \times Y$
Depends on a choice of metric g on Y, but only up to cobordism.

$\mathcal{M}_{\text{symp}}(g) :=$ moduli space of isolated holomorphic strips in M
Depends on a choice of metric g, but only up to cobordism.

If $\mathcal{M}_{\text{inst}}(g)$ and $\mathcal{M}_{\text{symp}}(g)$ are cobordant, then the conjecture would follow.

Motivating example

Motivating example

Theorem (Dostoglou-Salamon (1993))

If Y is a mapping torus, then $HF_{\mathrm{inst}}(Y) = HF_{\mathrm{symp}}(Y)$.

Motivating example

Theorem (Dostoglou-Salamon (1993))

If Y is a mapping torus, then $HF_{\mathrm{inst}}(Y) = HF_{\mathrm{symp}}(Y)$.

The proof shows that for a suitable g, there is a diffeomorphism $\mathcal{M}_{\mathrm{inst}}(g) \cong \mathcal{M}_{\mathrm{symp}}(g)$.

Motivating example

Theorem (Dostoglou-Salamon (1993))

If Y is a mapping torus, then $HF_{\mathrm{inst}}(Y) = HF_{\mathrm{symp}}(Y)$.

The proof shows that for a suitable g, there is a diffeomorphism $\mathcal{M}_{\mathrm{inst}}(g) \cong \mathcal{M}_{\mathrm{symp}}(g)$.

Key geometric input: If Y is a mapping torus, then $\mathcal{M}_{\mathrm{symp}}(g)$ can be taken to be the moduli space of holomorphic *cylinders* $\mathbb{R} \times S^1 \to M$ (as opposed to strips).

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

Motivating Example (cont'd)

Motivating Example (cont'd)

Restrict to the case $Y = S^1 \times \Sigma$, where Σ is a surface. Then

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

Motivating Example (cont'd)

Restrict to the case $Y = S^1 \times \Sigma$, where Σ is a surface. Then

- M is the moduli space of flat connections on Σ

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

Motivating Example (cont'd)

Restrict to the case $Y = S^1 \times \Sigma$, where Σ is a surface. Then

- M is the moduli space of flat connections on Σ (smooth by the assumptions on the bundle)

Motivating Example (cont'd)

Restrict to the case $Y = S^1 \times \Sigma$, where Σ is a surface. Then

- M is the moduli space of flat connections on Σ (smooth by the assumptions on the bundle)

- $\mathcal{M}_{\text{symp}}(g) = \left\{\text{isolated holomorphic cylinders } \mathbb{R} \times S^1 \to M\right\} / \mathbb{R}$

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

Motivating Example (cont'd)

Motivating Example (cont'd)

Each holomorphic cylinder lifts to a connection $A + \Phi\, ds + \Psi\, dt$ on $\mathbb{R} \times S^1 \times \Sigma$. This satisfies

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

Motivating Example (cont'd)

Each holomorphic cylinder lifts to a connection $A + \Phi\, ds + \Psi\, dt$ on $\mathbb{R} \times S^1 \times \Sigma$. This satisfies

$$\begin{aligned} \partial_s A + *\partial_t A &= d_A \Phi + *d_A \Psi \\ F_A &= 0 \end{aligned}$$

Motivating Example (cont'd)

Each holomorphic cylinder lifts to a connection $A + \Phi\, ds + \Psi\, dt$ on $\mathbb{R} \times S^1 \times \Sigma$. This satisfies

$$\begin{aligned} \partial_s A + *\partial_t A &= d_A \Phi + * d_A \Psi \\ F_A &= 0 \end{aligned}$$

The instanton equation on $\mathbb{R} \times S^1 \times \Sigma$ is:

$$\begin{aligned} \partial_s A + *\partial_t A &= d_A \Phi + * d_A \Psi \\ F_A &= *(\partial_s \Psi - \partial_t \Phi - [\Psi, \Phi]) \end{aligned}$$

Motivating Example (cont'd)

Each holomorphic cylinder lifts to a connection $A + \Phi \, ds + \Psi \, dt$ on $\mathbb{R} \times S^1 \times \Sigma$. This satisfies

$$\begin{aligned} \partial_s A + *\partial_t A &= d_A \Phi + * d_A \Psi \\ F_A &= 0 \end{aligned}$$

The instanton equation on $\mathbb{R} \times S^1 \times \Sigma$ is:

$$\begin{aligned} \partial_s A + *\partial_t A &= d_A \Phi + * d_A \Psi \\ F_A &= *(\partial_s \Psi - \partial_t \Phi - [\Psi, \Phi]) \end{aligned}$$

Dostoglou-Salamon show $*(\partial_s \Psi - \partial_t \Phi - [\Psi, \Phi])$ is small for a suitable metric g.

Motivating Example (cont'd)

Each holomorphic cylinder lifts to a connection $A + \Phi\, ds + \Psi\, dt$ on $\mathbb{R} \times S^1 \times \Sigma$. This satisfies

$$\begin{aligned} \partial_s A + *\partial_t A &= d_A \Phi + *d_A \Psi \\ F_A &= 0 \end{aligned}$$

The instanton equation on $\mathbb{R} \times S^1 \times \Sigma$ is:

$$\begin{aligned} \partial_s A + *\partial_t A &= d_A \Phi + *d_A \Psi \\ F_A &= *(\partial_s \Psi - \partial_t \Phi - [\Psi, \Phi]) \end{aligned}$$

Dostoglou-Salamon show $*(\partial_s \Psi - \partial_t \Phi - [\Psi, \Phi])$ is small for a suitable metric g.

Then they use an implicit function theorem to map holomorphic curves to instantons.

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

What makes the general case hard?

Strips can likely not be avoided, so it becomes a boundary-value problem.

I don't know how to make the Dostoglou-Salamon implicit function theorem work with this type of boundary-value problem.

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

What makes the general case hard?

Strips can likely not be avoided, so it becomes a boundary-value problem.

I don't know how to make the Dostoglou-Salamon implicit function theorem work with this type of boundary-value problem.

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

What makes the general case hard?

Strips can likely not be avoided, so it becomes a boundary-value problem.

I don't know how to make the Dostoglou-Salamon implicit function theorem work with this type of boundary-value problem.

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

A different approach

In the general situation, holomorphic strips still lift to connections on $\mathbb{R} \times Y$.

For suitable g, these connections have near-minimal Yang-Mills energy.

Instantons have minimal Yang-Mills energy (by definition).

Idea: Use the Yang-Mills heat flow to flow from (lifted) holomorphic strips to instantons.

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

A different approach

In the general situation, holomorphic strips still lift to connections on $\mathbb{R} \times Y$.

For suitable g, these connections have near-minimal Yang-Mills energy.

Instantons have minimal Yang-Mills energy (by definition).

Idea: Use the Yang-Mills heat flow to flow from (lifted) holomorphic strips to instantons.

A different approach

In the general situation, holomorphic strips still lift to connections on $\mathbb{R} \times Y$.

For suitable g, these connections have near-minimal Yang-Mills energy.

Instantons have minimal Yang-Mills energy (by definition).

Idea: Use the Yang-Mills heat flow to flow from (lifted) holomorphic strips to instantons.

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

A different approach

In the general situation, holomorphic strips still lift to connections on $\mathbb{R} \times Y$.

For suitable g, these connections have near-minimal Yang-Mills energy.

Instantons have minimal Yang-Mills energy (by definition).

Idea: Use the Yang-Mills heat flow to flow from (lifted) holomorphic strips to instantons.

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

A different approach

In the general situation, holomorphic strips still lift to connections on $\mathbb{R} \times Y$.

For suitable g, these connections have near-minimal Yang-Mills energy.

Instantons have minimal Yang-Mills energy (by definition).

Idea: Use the Yang-Mills heat flow to flow from (lifted) holomorphic strips to instantons.

A different approach (cont'd)

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

A different approach (cont'd)

Theorem (D. (2014))

There is a metric g on Y (and a holonomy perturbation) so that the (perturbed) Yang-Mills heat flow defines an injection

$$\mathcal{M}_{\mathrm{symp}}(g) \to \mathcal{M}_{\mathrm{inst}}(g).$$

A different approach (cont'd)

Theorem (D. (2014))

There is a metric g on Y (and a holonomy perturbation) so that the (perturbed) Yang-Mills heat flow defines an injection

$$\mathcal{M}_{\mathrm{symp}}(g) \to \mathcal{M}_{\mathrm{inst}}(g).$$

When Y is a mapping torus, this gives a new proof of Dostoglou-Salamon's result ($+$ some more techniques to get surjectivity)

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

Work in progress/future directions

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

Work in progress/future directions

- Surjectivity for general Y?

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

Work in progress/future directions

- Surjectivity for general Y?
- More general 4-manifolds?

Notation

The conjecture

How a proof might go

Motivating example

What makes the general case hard?

A different approach

Work in progress/future directions

Work in progress/future directions

- Surjectivity for general Y?
- More general 4-manifolds?

degree zero relative Donaldson invariants $\overset{?}{\cong}$ Katrin Wehrheim's degree zero relative quilt invariants

Work in progress/future directions

- Surjectivity for general Y?
- More general 4-manifolds?

$$\text{degree zero relative Donaldson invariants} \overset{?}{\cong} \text{Katrin Wehrheim's degree zero relative quilt invariants}$$

- Higher degree Donaldson/quilt invariants?

Thank you!