Bundle splittings on boundary-punctured disks

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C. Woodward (2020): Is there a similar result for strips $\mathbb{R} \times [0, 1]$?

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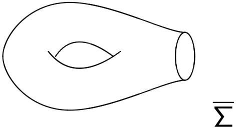

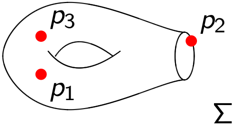

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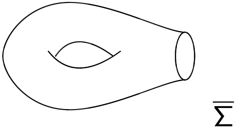

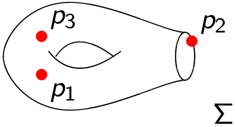

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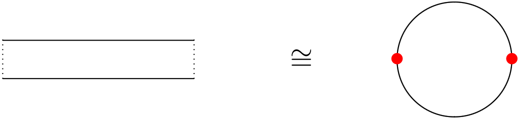

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- Then $(\mathcal{E}, R)$ is called a **(holomorphic) bundle pair** over Σ.

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Call bundle pairs $(\mathcal{E}, R)$ of this type **totally split**.

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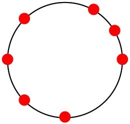
What I think: Yes (at least if R is transverse at the boundary punctures)

In fact, I think this holds for any simply-connected Σ:



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There is a correspondence

$$\mathcal{A}(E) \longleftrightarrow \mathcal{C}(E)$$

given by $A \mapsto \overline{\partial}_A := (d_A)^{0,1}$.

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3. Discuss progress made in the case where Σ has punctures.

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Grothendieck's result follows from these two corollaries. □

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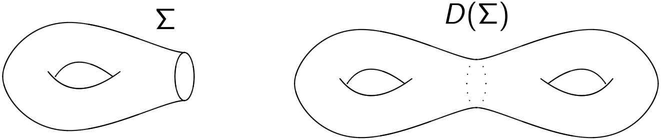

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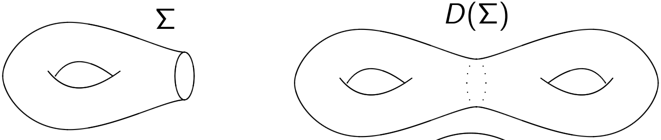
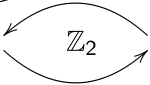


- The double has a natural $\mathbb{Z}_2$-action:



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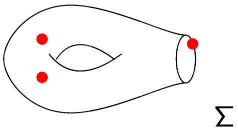

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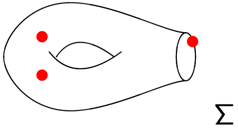

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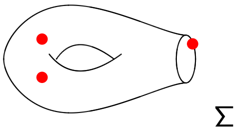

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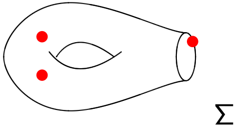

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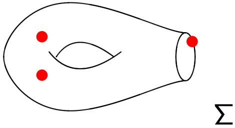

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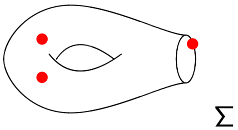

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  Also, many other analytic details are known (e.g., the expected Sobolev embedding and compactness results hold). These are due to Sibner–Sibner (1988), Kronheimer–Mrowka (1992), Råde (1995), Daskalopolous–Wentworth (1998), and others.

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Define $\mathcal{A}_\partial(E)$ to be the set of connections on $E \to \Sigma$ satisfying the above-discussed boundary and asymptotic holonomy conditions.

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This bundle automorphism satisfies the appropriate boundary and asymptotic conditions to imply that $(\mathcal{E}(A(\tau)), R)$ is isomorphic to $(\mathcal{E}(A_0), R)$ (cf. the homework from the beginning of the talk).

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... [work in progress]

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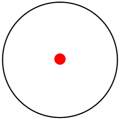

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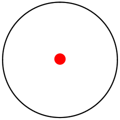

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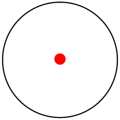

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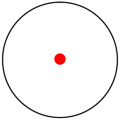

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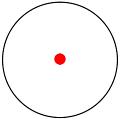

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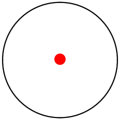

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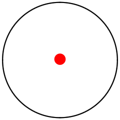

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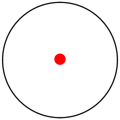

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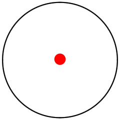

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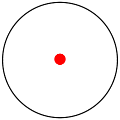

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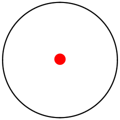
$$A_{\theta,0}(r) = \frac{1}{2} c_{\theta,0} r^2 + h_{\theta,0}, \qquad A_{\theta,k}(r) = h_{\theta,k} + ik \int_{r_0}^{r} A_{r,k}\, dr, \text{ for } k \neq 0$$

for some constants $h_{\theta,k}$, $c_{\theta,0}$.These constants have geometric meaning:

- $c_{\theta,0} = *F_A$
- The asymptotic holonomy of A around the puncture is $e^{-2\pi h_{\theta,0}}$.
- The holonomy around the boundary loop is equal to $-\pi c_{\theta,0} - 2\pi h_{\theta,0}$.
- On the bundle side, this holonomy recovers the equals the Maslov index $\mu(R)$ (times $i/2$), and so

$$\mu(R) = 2\pi i (c_{\theta,0} + 2 h_{\theta,0}).$$



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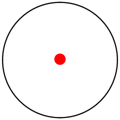
- The remaining data ($h_{\theta,k}$ and $A_{r,k}$) are constrained by the choice of R.



Thank you for your attention!