From instantons to quilts with seam degenerations

David L. Duncan

Michigan State University

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Notation

Σ; closed, connected, oriented surface with Riemannian metric

$P \to \Sigma$; principal $\mathrm{SO}(3)$-bundle

Assume P is non-trivial. Then there are no reducible flat connections.

(More generally, this entire discussion carries over with $\mathrm{PU}(r)$-bundles, $r \geq 2$, provided the degree is coprime to r.)

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Theorem (Dostoglou-Salamon (1993))

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$HF_{\text{inst}}(S^1 \times \Sigma)$ defined by counting (isolated) instantons on $\mathbb{R} \times S^1 \times \Sigma$

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The proof reduces to defining a bijection

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Summary of results

Replace $\mathbb{R} \times S^1 \times \Sigma$ with an oriented, connected 4-manifold Z with cylindrical ends (allowing the possibility of no ends).

Replace $\mathbb{R} \times S^1$ with an oriented surface S having the same number of cylindrical ends.

Assume there is a map $F : Z \to S$ that preserves the ends and has connected, non-empty fibers.

Assume there is an $\mathrm{SO}(3)$-bundle $R \to Z$ that restricts to the non-trivial bundle on some fiber of F.

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Assume for now that F is a fiber bundle.

The more general case will be discussed briefly at the end.

For $x \in S$, let $\Sigma_x = F^{-1}(x)$ be the fiber over x.

Let $M(Z)$ be the fiber bundle over S with fiber over $x \in S$ given by $M(\Sigma_x)$.

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Either use the Dostoglou-Salamon implicit function theorem, or use the Yang-Mills heat flow. (The latter generalizes to when F is not a fiber bundle.)

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Theorem (D.-McNamara (2014))

The map $\mathcal{N}$ is an injection.

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Corollary

The relative Donaldson invariant of (Z, R, F) agrees with the relative symplectic invariant of (Z, R, F).

Generalizes Dostoglou-Salamon's theorem, and gives an alternate proof.

'Relative' refers to the fact that the invariants take values in the Floer homology of the ends.

Should extend to polynomial invariants when Z is closed.

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- Discuss non-fiber bundle case.

This is where quilts enter the picture (e.g., the *quilted Atiyah-Floer conjecture*, and its 4-manifold generalizations).

Connections

$P \to \Sigma$ as above

$\mathcal{A}(\Sigma) = \{\text{connections on } P\}$

$\mathcal{A}_{\text{flat}}(\Sigma) = \{\alpha \in \mathcal{A}(\Sigma) \mid F_\alpha = 0\}$
where F_α is the curvature of α

$\mathcal{G}(\Sigma) = \{\text{gauge transformations on } P\}$

$\mathcal{G}_0(\Sigma) = \{g \in \mathcal{G}(\Sigma) \mid g \simeq \text{Id}\}$

The action of $\mathcal{G}_0(\Sigma)$ on $\mathcal{A}_{\text{flat}}(\Sigma)$ is free, and

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Complexified gauge group

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$G^{\mathbb{C}}$ complexification of $G = \mathrm{SO}(3)$

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These act on $\mathcal{A}(\Sigma)$ by identifying $\mathcal{A}(\Sigma)$ with the space of Cauchy-Riemann operators.

There is a preferred neighborhood $\mathcal{A}^{ss}(\Sigma)$ of $\mathcal{A}_{\mathrm{flat}}(\Sigma)$ on which $\mathcal{G}_0^{\mathbb{C}}$ acts freely.

There is some $\epsilon_0 > 0$ so that if $\|F_\alpha\|_{L^2} \leq \epsilon_0$, then $\alpha \in \mathcal{A}^{ss}(\Sigma)$.

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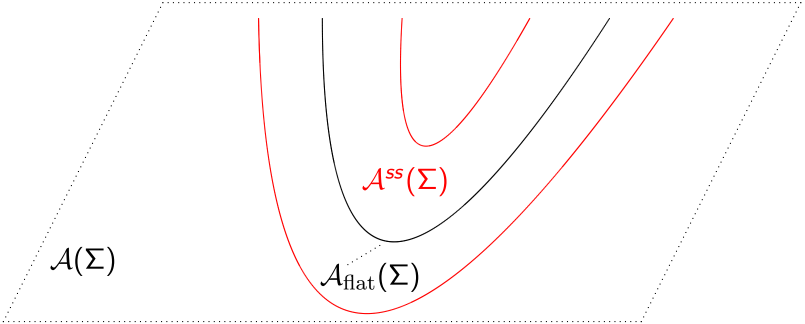
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Denote the projection by

$$\Pi : \mathcal{A}^{ss}(\Sigma) \longrightarrow M(\Sigma)$$

A property of Π

- The metric and orientation on Σ induce a Hodge star $*$
- $*^2 = -1$ on 1-forms
- $T_\alpha \mathcal{A}(\Sigma) = \{\text{Lie algebra-valued 1-forms}\}$

$\Rightarrow \mathcal{A}(\Sigma)$ has a complex structure given by $*$

Similarly, $M(\Sigma)$ has a complex structure J induced from $*$.

Π preserves this structure:

$$D_\alpha \Pi(*\mu) = J\, D_\alpha \Pi(\mu)$$

where $D_\alpha \Pi$ is the linearization at $\alpha \in \mathcal{A}^{ss}(\Sigma)$ and $\mu \in T_\alpha \mathcal{A}(\Sigma)$.

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- The metric and orientation on Σ induce a Hodge star $*$
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- The map Π takes holomorphic curves in $\mathcal{A}^{ss}(\Sigma)$ to holomorphic curves in $M(\Sigma)$.
- This can be strengthened as follows:

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Local picture: The case $S \times \Sigma$

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Suppose $A \in \mathcal{A}(S \times \Sigma)$ is a connection.

(s, t) coordinates on $S \subset \mathbb{C}$.

Write A in coordinates

$$A = \alpha + \phi\, ds + \psi\, dt,$$

$\alpha : S \to \mathcal{A}(\Sigma)$ defined by $\alpha(s,t) := A|_{\{(s,t)\} \times \Sigma}$

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A connection A on Z is an **(ASD) instanton** if

$$*_Z F_A = -F_A,$$

where $*_Z$ is the Hodge star on Z.

$A = \alpha + \phi\, ds + \psi\, dt$ is an instanton if and only if

$$\begin{aligned} \partial_s \alpha + *_\Sigma \partial_t \alpha &= d_\alpha \phi + *_\Sigma d_\alpha \psi \\ F_\alpha &= *_\Sigma (\partial_t \phi - \partial_s \psi - [\phi, \psi]) \end{aligned}$$

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In particular, if we can ensure the curvature F_α is small, then

$$u := \Pi(\alpha) : S \longrightarrow M(\Sigma)$$

is a well-defined function that is holomorphic.

We are starting to see a map

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Theorem (D. (2013))

Suppose S is closed or has cylindrical ends. Given any $\epsilon_0 > 0$, there is a metric on S so that if A is an instanton on Z with $\dim(A) \leq 3$, *then*

$$\sup_{x \in S} \|F_\alpha\|_{L^2(\Sigma_x)} \leq \epsilon_0.$$

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Define

$$\mathcal{N} : \{\text{instantons } A \text{ on } Z \text{ with } \dim(A) \leq 3\} / \{\text{gauge}\} \longrightarrow \{\text{holomorphic sections } S \to M(Z)\}.$$

by sending $[A]$ to the map $x \mapsto \Pi(\alpha(x))$.

Remarks

- The map $\mathcal{N}$ preserves the dimension of the moduli spaces.
- When S has cylindrical ends then $\mathcal{N}$ preserves the broken trajectory compactifications of the moduli spaces.

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Theorem (D.-McNamara (2014))

The map $\mathcal{N}$ is a $\mathcal{C}^1$-embedding.

Proof.

Injectivity of $\mathcal{N}$: If $\mathcal{N}([A_0]) = \mathcal{N}([A_1])$, then $\Pi(\alpha_0) = \Pi(\alpha_1)$ as maps $S \to M(Z)$. This implies

$$\alpha_0 = g^* \alpha_1$$

for some complex gauge transformation $g : S \to \mathcal{G}_0^{\mathbb{C}}$. The instanton equation and irreducibility force g to take values in the real gauge group $\mathcal{G}_0$. This implies A_0 and A_1 lie in the same gauge orbit, so $[A_0] = [A_1]$.

Injectivity of the linearization $D_{[A]}\mathcal{N}$ is similar.

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Corollary (Dostoglou-Salamon, D.-McNamara, D.-)

The map $\mathcal{N}$ is a diffeomorphism.

Proof 1 (Kähler case).

If Z is Kähler, then there is a complexified gauge group $\mathcal{G}(Z)^{\mathbb{C}}$. Any holomorphic section can be represented by a connection A_0 on Z that is in the semistable range. Then there is a (unique up to real gauge transformation) instanton A that is in the $\mathcal{G}(Z)^{\mathbb{C}}$-orbit of A_0. The map sending $[A_0]$ to $[A]$ is an approximate inverse of $\mathcal{N}$. □

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Proof 2 (general case).

Given a connection A_0 that represents a holomorphic section of $M(Z)$, one can use an implicit function theorem as in Dostoglou-Salamon [DS3] to find a nearby instanton A. This provides an approximate inverse to $\mathcal{N}$. □

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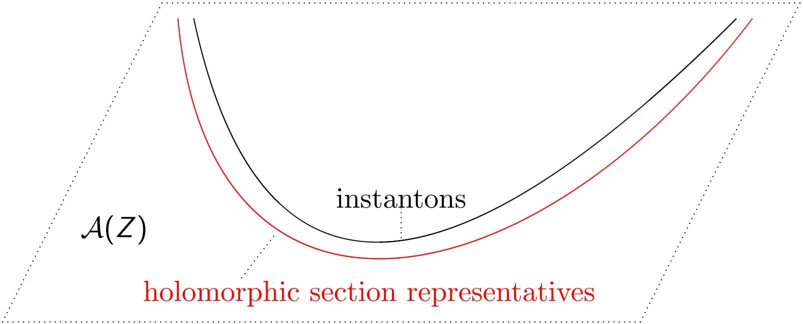
instantons
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More general 4-manifolds

Set-up (reminder)

Suppose

- S is a connected, oriented surface with cylindrical ends,
- Z is an oriented 4-manifold with the same number of cylindrical ends as S,
- $F : Z \to S$ preserves the ends and has connected, nonempty fibers, and
- $R \to Z$ is a principal $\mathrm{SO}(3)$-bundle that restricts to the non-trivial bundle on some fiber.

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Idea

Compare instantons on Z with certain symplectic objects, and try to extend the results above.

Instantons are well-defined, and the (relative) Donaldson invariant depends only on Z, the topological type of the bundle R, and the homotopy class of F.

The relevant symplectic objects are holomorphic *quilts*. These were defined by Wehrheim-Woodward [WW2], and one uses *Morse 2-functions* to obtain these from 4-manifolds.

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Morse 2-functions

$F : Z \to S$ is a **Morse 2-function** if it can be written locally as a 'homotopy of Morse functions'. That is, if each point in S lies in a neighborhood $I \times I \subset S$, with $I = [0, 1]$, and this neighborhood satisfies

- $F^{-1}(I \times I) = I \times Y$ where Y is a 3-manifold,
- in these coordinates

$$F(s, y) = (s, f_s(y)),$$

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Example 1: Suppose $F : Y \to S$ is a fiber bundle.

Example 2: Suppose Y is a closed 3-manifold and $f : Y \to S^1$ is a Morse function. Consider $Z = \mathbb{R} \times Y$ and $S = \mathbb{R} \times S^1$. Then $F(s, y) = (s, f(y))$ is a Morse 2-function. This is the **cylindrical case**.

In general, if Z has cylindrical ends, then we assume that any Morse 2-function restricts to the cylindrical case on the ends.

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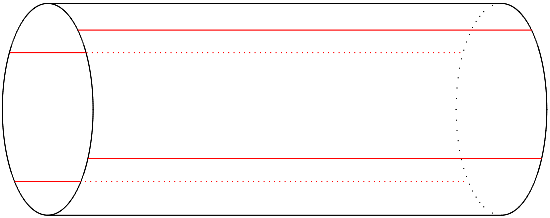
It is useful to trace out the critical values of the f_s in the surface S. This is called the **Cerf graphic**. The critical values form the **seams**, and the regular values form the **patches**.

Morse 2-functions

In the cylindrical case $Z = \mathbb{R} \times Y$, the Cerf graphic takes the form:



$$S = \mathbb{R} \times S^1$$

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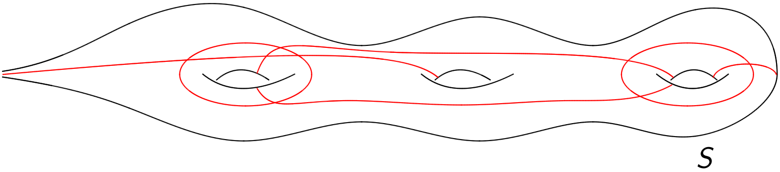
Here is the Cerf graphic for some Morse 2-function $F : Z \to S$, where S is a punctured genus 3 surface.



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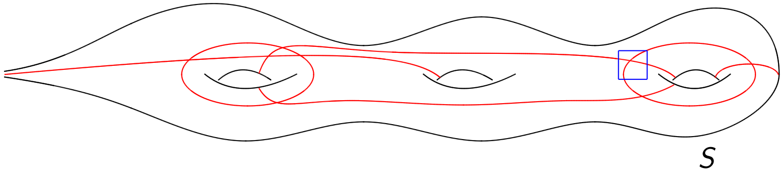

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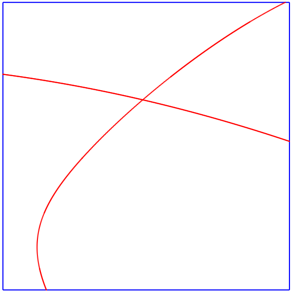

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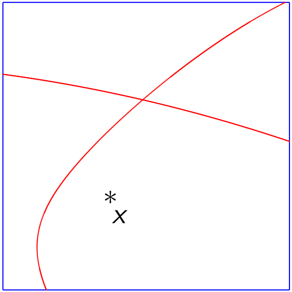

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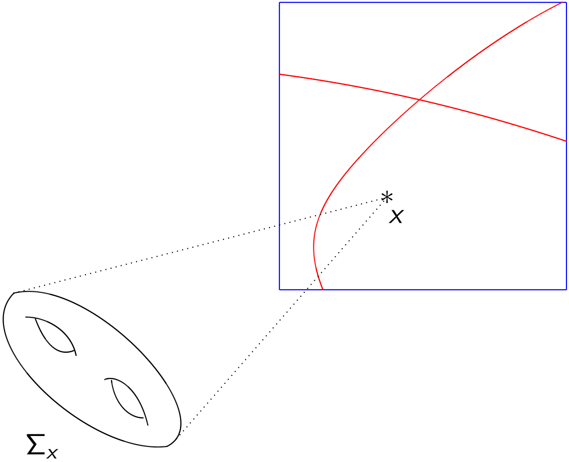

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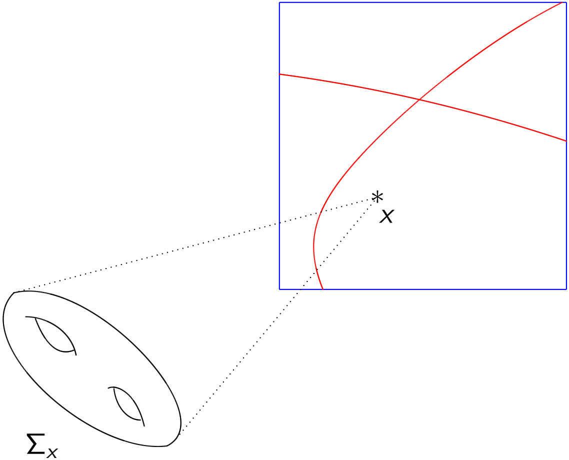

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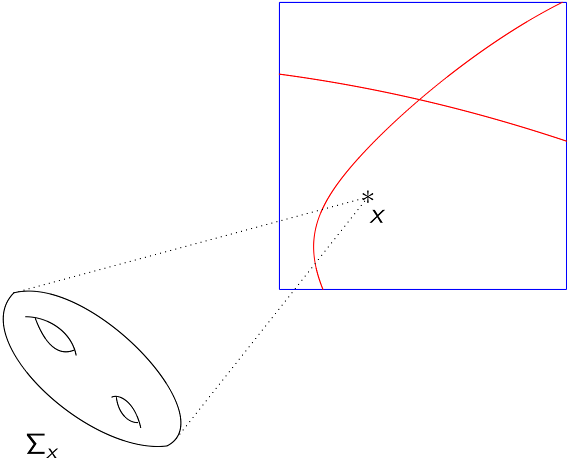

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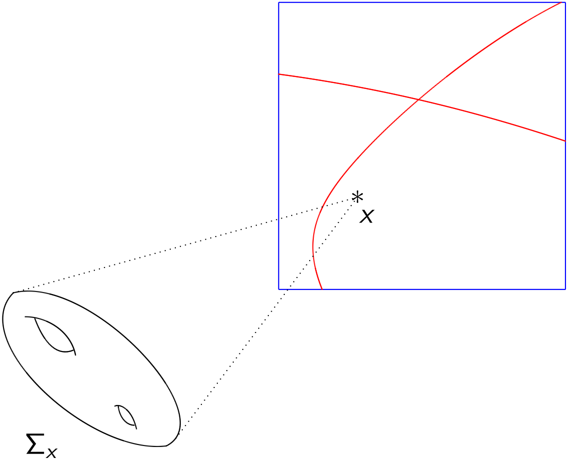

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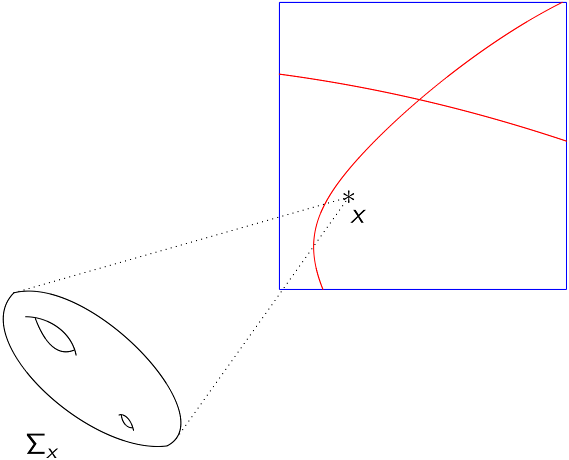

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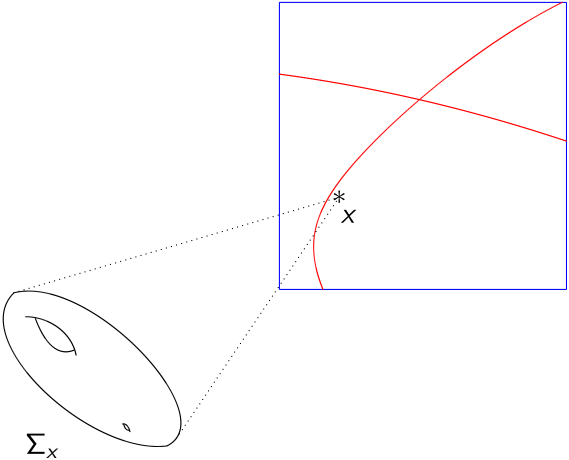

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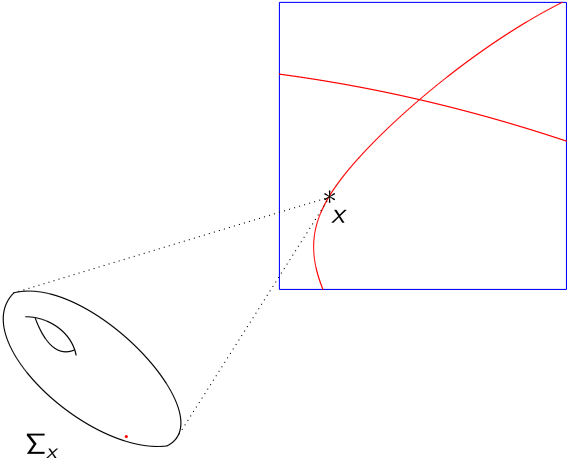

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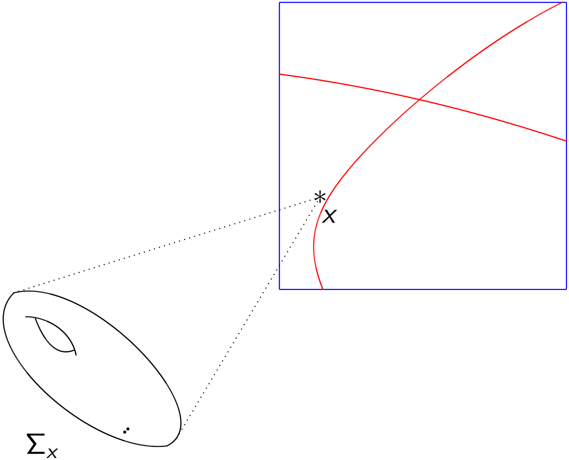

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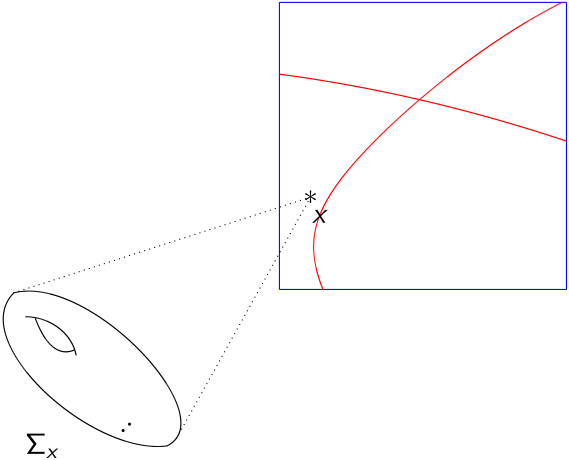

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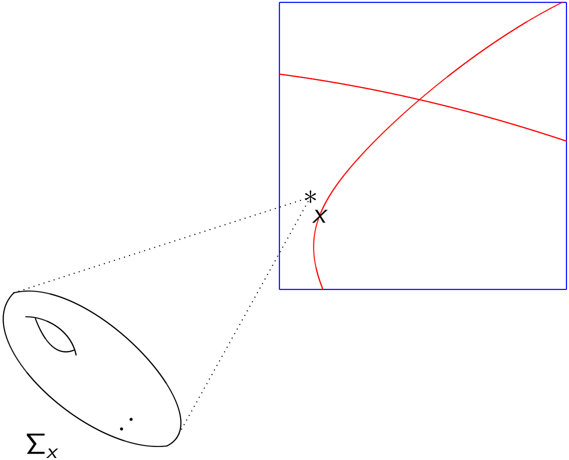

Morse 2-functions

Theorem (Gay-Kirby (2011))

Suppose $F : Z \to S$ is any smooth function with connected and non-empty fibers. Then F is homotopic to a Morse 2-function that has connected and non-empty fibers. This Morse 2-function is unique up to certain homotopies of Morse 2-functions.

Quilts from Morse 2-functions

The treatment here follows Wehrheim-Woodward [WW1] and Wehrheim [W].

Assume $F : Z \to S$ is a Morse 2-function.

Consider the surface S with the diagram coming from the critical points.

Each point x in a patch is a regular point for F. The inverse image $F^{-1}(x) = \Sigma_x$ is therefore a closed, connected, oriented surface. Label this point with the symplectic manifold $M(\Sigma_x)$.

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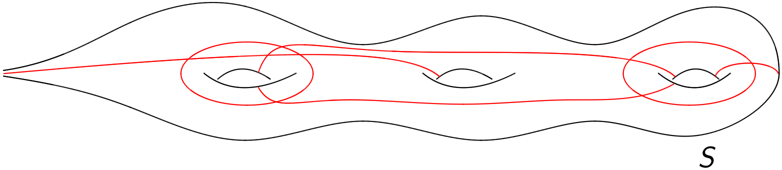

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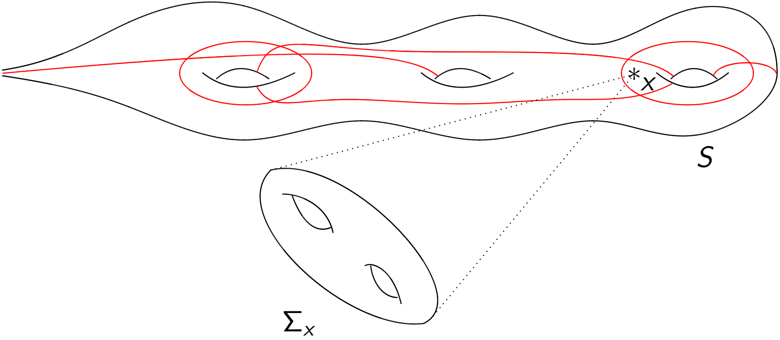

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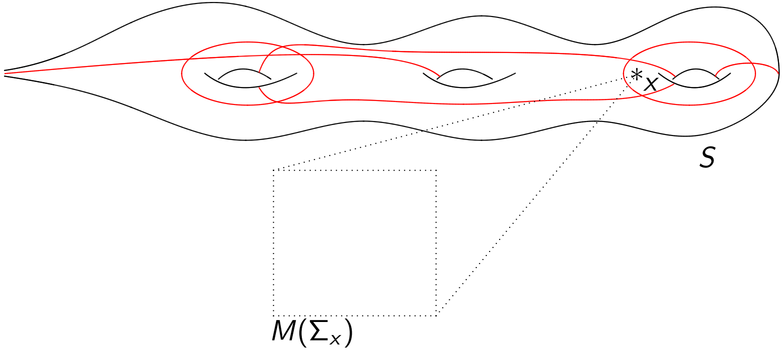

Quilts from Morse 2-functions

Over each patch S_i is a fiber bundle, as before: $M(Z_i) \to S_i$

Think of the singular fibers as interpolating between non-singular fibers. (In symplectic language, there is a *Lagrangian correspondence* here.)

A **quilt** is a collection $\{u_i\}$ where u_i is a section $S_i \to M(Z_i)$, and we require that these agree on the seams.

The quilt is **holomorphic** if each u_i is holomorphic with respect to some complex structure.

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Quilts from Morse 2-functions

Wehrheim-Woodward show the following:

There is a complex structure on each $M(Z_i)$ such that, after perturbing, the moduli space of holomorphic quilts is smooth and finite-dimensional.

Counting the elements of the zero-dimensional component of this moduli space defines the relevant symplectic invariant.

In the cylindrical case, this invariant depends only on $\mathbb{R} \times Y$, the isomorphism class of R, and the homotopy class of f.

(For more general 4-manifolds, it is not yet known whether this depends only on the homotopy class of F. Establishing this is an active research project of Wehrheim.)

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From instantons to quilts

Suppose A is an instanton. For each patch S_i and $x \in S_i$, set $\alpha_i(x) = A|_{\{F^{-1}(x)\}}$.

As before, there is a metric on Z for which each $\alpha_i(x)$ has small curvature.

Define $u_i(x) := \Pi(\alpha_i(x))$. So u_i is a section $S_i \to M(Z_i)$.

Then the fiber bundle argument from above shows that u_i is holomorphic in the interior of the patch S_i.

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That is, just as before, we should get a map from the moduli space of instantons to the moduli space of holomorphic quilts.

The catch is that the relevant metric on each fiber Σ_x becomes singular (blows up) as x approaches a seam. This reflects the fact that the fibers over the seams are singular.

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Questions/Future directions

Q1: Do the u_i actually have the expected boundary conditions?

This should be the case. The work is in studying how the map $\Pi : \mathcal{A}^{ss}(\Sigma) \to M(\Sigma)$ behaves under a degeneration of the metric.

If this is the case, then the relevant holomorphic quilts would be holomorphic with respect to a complex structure that *becomes singular near the boundary*.

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Questions/Future directions

Q2: Does the associated moduli space have the regularity/compactness properties needed to obtain a well-defined invariant?

Questions/Future directions

Supposing these questions have favorable answers, then we should be able to extend the analysis from the fiber bundle picture above to show that the induced map $\mathcal{N}$ is a diffeomorphism from the moduli space of instantons to the moduli space of holomorphic quilts.

Even if these questions do not have favorable answers, there is already enough going on here to obtain strong results. For example, these ideas were used to prove a compactness theorem [D1] that studies the behavior of instantons under certain degenerations of the underlying metric.

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Thank you for your attention.

Bibliography

[DS2] S. Dostoglou, D. Salamon. Instanton homology and symplectic fixed points. *Symplectic geometry*, 57-93, *London Math. Soc. Lecture Note Ser.*, 192, Cambridge Univ. Press, Cambridge, 1993.

[DS3] S. Dostoglou, D. Salamon. Self-dual instantons and holomorphic curves. *Ann. of Math.* (2) 139 (1994), no. 3, 581-640.

[D1] D. Duncan. Compactness results for the quilted Atiyah-Floer conjecture. arXiv:1212.1547v3.

[D2] D. Duncan. Relative Donaldson and symplectic invariants for 4-manifolds. In preparation.

[GK] D. Gay, R. Kirby. Indefinite Morse 2-functions: broken fibrations and generalizations, 2011. arXiv:1102.0750.

Bibliography

[N] T. Nishinou, Convergence of adiabatic family of anti-self-dual connections on products of Riemann surfaces. *Journal of Mathematical Physics*, 51, 022306 (2010); doi: 10.1063/1.3318164

[W] K. Wehrheim. Outline of compactness proofs for Atiyah-Floer degenerations in the absence of reducibles, private communication, and talks at Columbia, UW Madison, ETH Zurich. 2004-6.

[WW1] K. Wehrheim, C. Woodward. Floer field theory. Preprint.

[WW2] K. Wehrheim, C. Woodward. Quilted Floer cohomology. *Geometry and Topology*, 14:833-902, 2010. arXiv:0905.1370.