Classifying Fixed Points of Automorphisms on Character Varieties

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Abstract

A commonly recurring theme in mathematics is being able to tell if two objects are distinct, up to some equivalence. For example, in an algebraic context we can tell if two groups are different if one is Abelian and the other is not. Properties such as these are called invariants. One such invariant of manifolds is the character variety, which has been widely studied. When considering a specific class of manifolds called mapping tori, one can instead study the fixed points of the induced automorphism on a character variety of the fiber. This paper explores these fixed points when the fiber is the genus 1 surface and also the punctured genus 2 surface. For the genus 1 surface, we consider any automorphism induced by an orientation-preserving homeomorphism. For the punctured genus 2 surface, we consider a specific infinite family of maps described by McCullough and Miller in [1].

1 Introduction

Let G be a compact Lie group. The G-character variety is an invariant of manifolds; this means that given two manifolds they are distinct if they have distinct G-character varieties. One interesting family of manifolds is the mapping torus of a surface. Given a surface Σ and an orientation-preserving homeomorphism $\phi : \Sigma \to \Sigma$, the mapping torus of ϕ given Σ is defined as $M_\phi = \frac{\Sigma \times [0,1]}{\sim}$ where the equivalence is given by $(\phi(x), 0) \sim (x, 1)$. The reason why this class of 3-manifolds is so nice is that the character variety of M_ϕ maps surjectively onto the fixed point set of the automorphism Φ of $\chi(\Sigma, G)$, where Φ is defined as

$$\Phi([f]) = [f \circ \phi_*]. \tag{1}$$

See Lemma 5.4 [3]. Here $\phi_* : \pi_1(\Sigma) \to \pi_1(\Sigma)$ is the isomorphism induced by ϕ.

In section 3, we consider $\Sigma = T^2$, the genus 1 surface, and an arbitrary ϕ. In this case, we give a complete characterization of the fixed point set of Φ for G equals U(n) and SU(n); see Theorem 3.2. This characterization also appears in Kirk–Klassen [2] and Jeffrey [3]. The proof of Theorem 3.2 essentially reduces to linear algebra because $\pi_1(T^2)$ is Abelian.

In section 4, we move on to the case of the punctured genus 2 surface, which we denote $\Sigma = T^2 \# T^2 \setminus \{x_0\}$. Since Σ is connected we can choose x_0 arbitrarily. In this case we lose the commutativity of the image of some element of our character variety. Due to the complexity this adds, instead of an arbitrary ϕ, we consider an infinite family of ϕ analyzed by McCullough and Miller in [1]. The broader relevance of this family is that it generates a subgroup of the genus 2 Torelli group that is not finitely generated. We give a complete description of the fixed point set for every element of this family, and find that each fixed point set is connected.

2 Definitions

Let G be a compact Lie group, Σ a surface. We will define the *G-representation variety* of Σ by

$$\mathrm{R}(\Sigma, \mathrm{G}) = \hom(\pi_1(\Sigma), \mathrm{G})$$

and the *G-character variety* of Σ by

$$\chi(\Sigma, \mathrm{G}) = \frac{\mathrm{R}(\Sigma, \mathrm{G})}{\mathrm{G}}$$

where the G action is given by conjugation. Specifically, given $f, f' \in R(\Sigma, \mathrm{G})$ we say that $f \sim f'$ if and only if there exists some $M \in \mathrm{G}$ such that

$$f(x) = M f'(x) M^{-1}$$

for all $x \in \pi_1(\Sigma)$. Due to the fact that a homomorphism is completely determined by how it acts on the generating set, given $\pi_1(\Sigma)$ is generated by $\{g_1, g_2, \ldots, g_n\}$ we will denote $[f] \in \chi(\Sigma, \mathrm{G})$ by $[f(g_1), f(g_2), \ldots, f(g_n)]$, where the equivalence is given by simultaneous conjugation of all of the elements.

3 Fixed points in genus 1

3.1 G equals U(n) or SU(n)

Let G be U(n) or SU(n) for some n, and $\Sigma = T^2$. Denote α, β to be standard generators of $\pi_1(T^2)$. Let $\phi : T^2 \to T^2$ be an orientation-preserving homeomorphism, with the associated pullback defined by the monodromy matrix,

$$\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$$

Due to the fact that ϕ is orientation-preserving we have that this matrix is in $\mathrm{SL}_2(\mathbb{Z})$. Let $\Phi : \chi(T^2, G) \to \chi(T^2, G)$ be the automorphism of the character variety induced by ϕ. Recall, given $[f] \in \chi(T^2, G)$ we denote $[f(\alpha), f(\beta)]$ by $[A, B] \in \frac{\mathrm{SU(2)}^2}{\mathrm{SU(2)}}$.

Lemma 3.1. *$[A, B] \in \chi(T^2, G)$ is a fixed point of Φ if and only if $[A^{k_1}B^{k_2}, A^{k_3}B^{k_4}] = [A, B]$*

Proof. Let $[A, B] \in \chi(T^2, G)$, Φ be as defined above. Then $[A, B]$ is a fixed point under Φ if and only if $\Phi([A, B]) = [A, B]$. Using the monodromy matrix above we can see that the $\Phi([A, B]) = [A^{k_1}B^{k_2}, A^{k_3}B^{k_4}]$. □

By definition of the character variety we have that $[A, B]$ is a fixed point of Φ if and only if there exists some $M \in G$ such that,

$$MAM^{-1} = A^{k_1}B^{k_2}$$

$$MBM^{-1} = A^{k_3}B^{k_4}$$

Since G equals U(n) or SU(n), we can assume that both A and B are diagonal, as the image of ϕ_* must be an Abelian subgroup of G, and we know that any pair of commuting unitary matrices are simultaneously diagonalizable. So we will choose a representative $(A, B) \in [A, B]$ where both A and B are diagonal; namely, we will say $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ and similarly $B = \mathrm{diag}(b_1, b_2, \ldots, b_n)$. We will here use the fact that A, B are diagonal to note that $A^{k_1}B^{k_2}$ and $A^{k_3}B^{k_4}$ must both also be diagonal. So we only need to consider diagonal matrices which are similar to A and B. Due to the fact that similar matrices share eigenvalues, any diagonal matrices similar to A and B must just be permutations of the diagonal elements of A and B, respectively. Then we now have that $[A, B]$ is a fixed point of Φ if and only if there exists some $\sigma \in S_n$, the permutation group on n elements, such that,

$$a_{\sigma(i)} = a_i^{k_1} b_i^{k_2}$$

$$b_{\sigma(i)} = a_i^{k_3} b_i^{k_4}$$

for all i in $\{1, 2, \ldots, n\}$.

Consider the substitution $a_i = e^{2\pi i x_i}$ and $b_i = e^{2\pi i y_i}$. Here we will assume that $x_i, y_i \in [0, 1)$ so that each is uniquely determined by A, B up to a reordering of the components. Then we can use the definition of the complex logarithm to turn the above into the following; $[A, B]$ is a fixed point if and only if there exists some $\sigma \in S_n$, $c_i, d_i \in \mathbb{Z}$ such that

$$x_{\sigma(i)} = k_1 x_i + k_2 y_i + c_i$$

$$y_{\sigma(i)} = k_3 x_i + k_4 y_i + d_i$$

for each i in $\{1, 2, \ldots, n\}$. We can further simplify this by just quantifying over the σ, and having the conditions

$$k_1 x_i + k_2 y_i - x_{\sigma(i)} \in \mathbb{Z}$$

$$k_3 x_i + k_4 y_i - y_{\sigma(i)} \in \mathbb{Z}$$

with i in $\{1, \ldots, n\}$. Let I_n be the $n \times n$ identity matrix, and define

$$\Omega = \left[\begin{array}{c|c} k_1 I_n & k_2 I_n \\ \hline k_3 I_n & k_4 I_n \end{array}\right]$$

Let $\sigma \in S_n$, and let $e_1, e_2, \ldots, e_n$ be the canonical orthonormal basis for $\mathbb{R}^n$. Define the permutation matrix M_σ as

$$M_\sigma = \begin{bmatrix} e_{\sigma(1)} & e_{\sigma(2)} & \cdots & e_{\sigma(n)} \end{bmatrix}$$

We are viewing our e_i as column vectors, so M_σ is an $n \times n$ integer matrix. Define the block matrix

$$P_\sigma = \left[\begin{array}{c|c} M_\sigma & 0 \\ \hline 0 & M_\sigma \end{array}\right],$$

then we can rephrase the above to be that $[A, B]$ is a fixed point if and only if there exists some permutation σ such that

$$(\Omega - P_\sigma)\vec{v} \in \mathbb{Z}^{2n}$$

where $\vec{v} = [x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]^T$. In summary, we have the following.

Theorem 3.2. *Given ϕ defined as above, G = U(n) for some n, and Φ the induced map of ϕ. Choose the representatives of $[A, B]$ such that A = diag$(e^{2\pi i x_1}, e^{2\pi i x_2}, \ldots, e^{2\pi i x_n})$ and B = diag$(e^{2\pi i y_1}, e^{2\pi i y_2}, \ldots, e^{2\pi i y_n})$. Then $[A, B]$ is a fixed point of Φ if and only if*

$$(\Omega - P_\sigma)\vec{v} \in \mathbb{Z}^{2n}$$

where $\Omega, P_\sigma, \vec{v}$ are as defined above.

This extends to U(n) Proposition 5.5 of [2] and is a special case of a result by Jeffrey Lemma 5.10 [3].

3.2 G equals SU(2)

We will look at how this analysis applies to the specific case G=SU(2), and $k_1 + k_4 \neq \pm 2$. Then the fixed points can be represented as points on the square $[0, 1]^2 \subset \mathbb{R}^2$, namely the pairs (x_1, y_1) as defined above. Setting $x = x_1$ and $y = y_1$, we have $[A, B]$ is a fixed point if and only if

$$(k_1 - 1)x + k_2 y = c \qquad\qquad (k_1 + 1)x + k_2 y = c$$
$$\text{or}$$
$$k_3 x + (k_4 - 1)y = d \qquad\qquad k_3 x + (k_4 + 1)y = d$$

for some $\delta, \gamma \in \mathbb{Z}$. We can solve for x, y to get the matrix equations

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2 - k_1 - k_4} \begin{bmatrix} k_4 - 1 & -k_2 \\ -k_3 & k_1 - 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \tag{2}$$

in the first case or

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2 + k_1 + k_4} \begin{bmatrix} k_4 + 1 & -k_2 \\ -k_3 & k_1 + 1 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} \tag{3}$$

in the second case. We can see that in this application we have that the fixed point set is a finite set, and so it is a collection of isolated points.

Example 1. Consider the case $k_1 = 858, k_2 = 598, k_3 = -260, k_4 = -181$. The images in Figure 1 and Figure 2 are representations of T^2; T^2 is obtained by gluing the opposite edges of the square together. Then T^2 is a branched double cover of the character variety. We generate these images by letting c, d range over integer values. We find that there is only a finite number of c, d we need to consider due to the repeating nature of the representation of T^2.

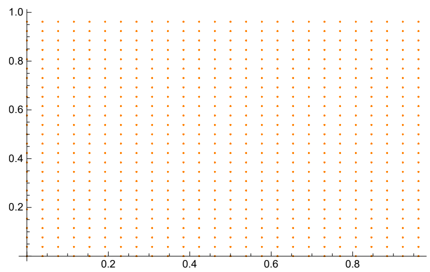

Figure 1: The solution set for equation (2) above.

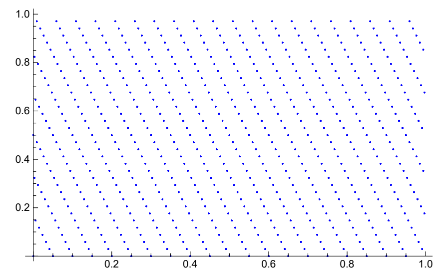

Figure 2: The solution set for equation (3) above.

4 Fixed points in genus 2

In this section we consider the cases $\Sigma = T^2 \# T^2 \setminus \{x_0\}$, and $G = SU(2)$. We are going to consider homeomorphisms given by Dehn twists around various curves on Σ.

Let the standard generating set of $\pi_1(\Sigma)$ be given by $\{\alpha_1, \beta_1, \alpha_2, \beta_2\}$, see Figure 3 on the left. In order to make computations easier we will be considering $\pi_1(\Sigma)$ under a nonstandard generating set. Namely, we will be using the generating set $\{\alpha, \beta, \gamma, \delta\}$ on the right of Figure 3.

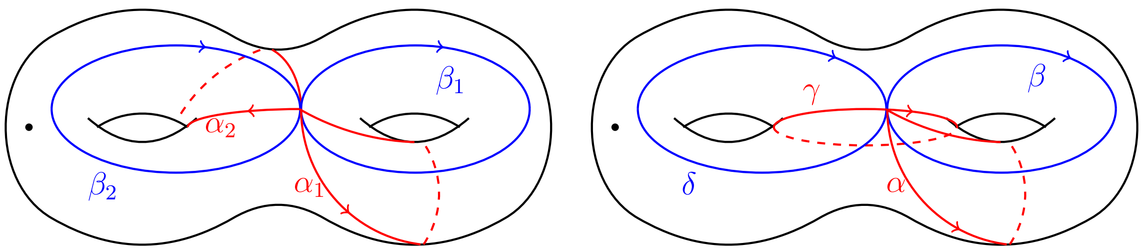

Figure 3: Illustrated here are the standard generating set for $\pi_1(\Sigma)$, as well as our nonstandard generating set.

Our nonstandard generating set can also be described algebraically:

$$\alpha = \alpha_1 \tag{4}$$

$$\beta = \beta_1 \tag{5}$$

$$\gamma = (\beta_2 \alpha_2 \alpha_1)^{-1} \tag{6}$$

$$\delta = \beta_2 \tag{7}$$

This choice of basis allows us to work more easily with the curves described by [1]. In the sections that follow we will consider a family of homeomorphisms and describe the fixed point sets of their induced maps on the character variety of Σ. We will denote

$$\Gamma(m_1, m_2) = \begin{bmatrix} m_1 & -\overline{m}_2 \\ m_2 & \overline{m}_1 \end{bmatrix} \tag{8}$$

for any matrix in SU(2). We know that any matrix in SU(2) can be written as $\Gamma(m_1, m_2)$ for some $m_1, m_2 \in \mathbb{C}$.

4.1 Curve with no loops

Let τ_0 be the curve defined in Figure 4, viewed as a curve in Σ.

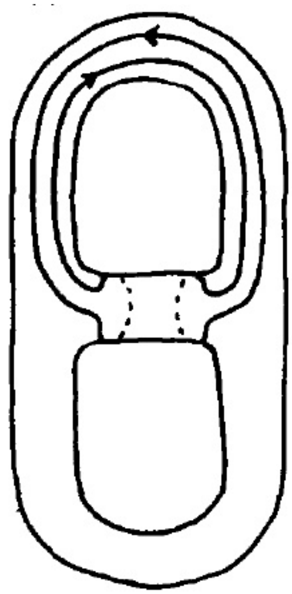

Figure 4: Illustrated here is a curve in $T^2 \# T^2$, view this as a curve τ_0 in Σ. Figure courtesy of [1, Fig. 3(a)].

Define $\phi_0 : \Sigma \to \Sigma$ to be the Dehn twist induced by τ_0. So ϕ_{0*} is the induced map on the fundamental group, and Φ_0 is the induced map on the character variety, as in equation (1).

Lemma 4.1. *In terms of the generating set $\{\alpha, \beta, \gamma, \delta\}$ defined in section 1,*

$$\phi_{0*}(\alpha) = [\beta^{-1}, \gamma^{-1}]\alpha[\gamma, \beta^{-1}]$$

$$\phi_{0*}(\beta) = \beta$$

$$\phi_{0*}(\gamma) = \gamma$$

$$\phi_{0*}(\delta) = [\beta^{-1}, \gamma^{-1}]\delta[\gamma^{-1}, \beta]$$

Proof. As described in [4] the action of ϕ_{0*} on $x \in \pi_1(\Sigma)$ is determined by how τ_0 intersects with a representative of x. For this reason we can see that there are representatives of b and c that do not intersect τ_0 anywhere, and so b and c stay fixed under ϕ_{0*}. This is one of the primary reasons we have chosen this generating set. When considering the image of a and d we note that each intersects τ_0 twice, and due to the location they intersect we get the claimed permutations of the commutator $[\beta, \gamma]$. $\square$

In order to find the fixed points of this map we will have to fix certain notation. We will denote $A = \Gamma(a_1, a_2)$, $B = \Gamma(b_1, b_2)$, $C = \Gamma(c_1, c_2)$, and $D = \Gamma(d_1, d_2)$ using the same labelling defined in (8). Due to the fact that every element of SU(2) is diagonalizable, we know that there is some representative of $[A, B, C, D]$ where C is diagonal; we will be working with this representative. Then $[A, B, C, D]$ is a fixed point if and only if

$$[A, B, C, D] = [\Phi_0(A), \Phi_0(B), \Phi_0(C), \Phi_0(D)]$$

with

$$\Phi_0(A) = [B^{-1}, C^{-1}]A[C, B^{-1}] \tag{9}$$

$$\Phi_0(B) = B \tag{10}$$

$$\Phi_0(C) = C \tag{11}$$

$$\Phi_0(D) = [B^{-1}, C^{-1}]D[C^{-1}, B] \tag{12}$$

We will compute all fixed points of Φ_0 by breaking up the possibilities for B and C into different cases. Specifically, given $B = \Gamma(b_1, b_2)$ and $C = \Gamma(c, 0)$ we will be considering the cases $b_1 = 0$, $b_2 = 0$, and $b_1, b_2 \neq 0$ and $c^2 = 1$, $c^2 = -1$, and $c^4 \neq 1$. First we need a lemma so that we know that we do not have to deal with the conjugation action.

Lemma 4.2. *Given $A, B \in SU(2) \setminus \{\pm Id\}$ there exists an $M \in SU(2) \setminus \{\pm Id\}$ such that*

$$A = MAM^{-1}$$

$$B = MBM^{-1}$$

if and only if $A \in C_B$, where C_B is the centralizer of B.

Proof. The centralizer of a matrix is determined by a great circle which goes through $\{\pm Id\}$, and so is uniquely determined by some non-identity element on it. In this case we know both centralizers contain M, and so both must be the same. In the other direction, if $A \in C_B$ then we can use $M = B$ to satisfy the identity. □

Notice that we have exactly this statement in equations (1), (2) above. So if $[B, C] \neq Id$ then we do not need to consider the conjugation action. We are now ready to consider the various possible cases.

Case 1. $[B, C] = \pm Id$

In this case we have that $[B^{-1}, C^{-1}] = [B^{-1}, C] = [B, C^{-1}] = \pm Id$, and so any A, D will satisfy equations (1) and (4) above. So we have our first two sets of fixed points

$$U_1 = \{(A, B, C, D) | [B, C] = Id\}$$

$$U_2 = \{(A, B, C, D) | [B, C] = -Id\}$$

Notice that this case handles six of the nine possible cases, namely $C = \pm Id$ and any possible B, $B = \Gamma(b, 0)$ and any possible C, and $B = \Gamma(0, b)$ with $C = \Gamma(\pm i, 0)$. So we only need to consider three additional cases.

Case 2. $B = \Gamma(0, b)$ *and* $C = \Gamma(c, 0)$ *with* $c^4 \neq 1$

Given (1) we know that A is a fixed point if and only if $A[B^{-1}, C] = [B^{-1}, C^{-1}]A$. Computing both of those we get that $A[B^{-1}, C] = \Gamma(\frac{a_1}{c^2}, \frac{a_2}{c^2})$ and $[B^{-1}, C^{-1}]A = \Gamma(a_1 c^2, \frac{a_2}{c^2})$. So A is a fixed point iff $a_1 = 0$, as we are assuming $c^4 \neq 1$. Doing a similar computation for D given (4) we have $D[B, C^{-1}] = \Gamma(c^2 d_1, c^2 d_2)$ and $[B^{-1}, C^{-1}]D = \Gamma(c^2 d_1, \frac{d_2}{c^2})$. So in this case we get that $d_2 = 0$. Here we arrive at our third set of fixed points

$$U_3 = \{(\Gamma(0,a), \Gamma(0,b), \Gamma(c,0), \Gamma(d,0)) | a,b,c,d \in S^1, c^4 \neq 1\}$$

We will handle the other two cases at once.

Case 3. $B = \Gamma(b_1, b_2)$ *and* $C = \Gamma(c, 0)$ *with* $b_1, b_2 \neq 0, c^2 \neq 1$

In this case, as should be expected, the computations are much nastier. We get that

$$A[B^{-1}, C] = \Gamma(a_1 + (1 - c^{-2})(a_1 b_2 c^2 + a_2 \overline{b_1})\overline{b_2}, a_1 b_1 b_2 (c^2 - 1) + a_2 |b_1|^2 + a_2 |b_2|^2 c^{-2}) \quad (13)$$

$$[B^{-1}, C^{-1}]A = \Gamma(\frac{b_1 b_2 (c^2 - 1)\overline{a_2} + a_1(|b_1|^2 c^2 + |b_2|^2)}{c^2}, b_1 b_2 \overline{a_1}(c^{-2} - 1) + a_2 |b_1|^2 + \frac{a_2 |b_2|^2}{c^2}) \quad (14)$$

First we will consider the second component. We are assuming $b_1, b_2 \neq 0, c^2 \neq 1$ so equality of those components simplifies to $a_1(c^2 - 1) = \overline{a_1}(c^{-2} - 1)$. We can then use the fact that $\frac{\overline{a_1}}{a_1} = \frac{|a_1|^2}{a_1^2}$ to simplify this to $a_1 = \pm i \frac{\sqrt{1 - |a_2|^2}}{c}$. Using the substitution $c = \pm i \frac{|a_1|}{a_1}$ in both of the first components we get that $(a_1 + \overline{a_1})(-b_1 b_2 \overline{a_2} + (a_1 b_2 - b_2 \overline{a_1} + a_2 \overline{b_1})\overline{b_2}) = 0$. This implies that either $\text{Re}(a_1) = 0$ or $2i\text{Im}(b_1 b_2 \overline{a_2}) + a_1 |b_2|^2 = 0$. Note that

$$\text{Re}(2i\text{Im}(b_1 b_2 \overline{a_2}) + a_1 |b_2|^2) = 0 + |b_2|^2 \text{Re}(a_1),$$

and so if $\text{Re}(a_1) \neq 0$ the second condition can never be fulfilled, and so is redundant and can be ignored. Notice that if $\text{Re}(a_1) = 0$ then we have that $\text{Im}(c) = 0$ or $a_1 = 0$. We have by assumption that $\text{Im}(c) \neq 0$, so it must be the case that $a_1 = 0$. So A is forced to be off diagonal.

Moving to D we have that

$$D[B,C^{-1}] = \Gamma(d_1 + (c^2-1)(|b_2|^2 d_1 + b_2\overline{b_1 d_2}), d_2 + |b_2|^2(c^2-1) - b_2\overline{b_1}(c^2-1)\overline{d_1}) \quad (15)$$

$$[B^{-1},C^{-1}]D = \Gamma(d_1 + (1-c^{-2})(b_2 c^2 d_1 + d_2\overline{b_1})\overline{b_2}, b_1 b_2(c^2-1)d_1 + |b_1|^2 d_2 + |b_2|^2 d_2 c^{-2}) \quad (16)$$

If $d_2 = 0$ then we can compute that $D = \Gamma(\pm i\frac{|b_1|}{b_1}, 0)$. Assuming this is not the case, the first component here gives us that $b_2\overline{d_2}c^2 = d_2\overline{b_2}$, and using that we can find that the second gives us the two cases, either $\overline{b_2}d_2 = b_2\overline{d_2}$ or $\overline{b_2}d_2 + b_2\overline{d_2} = b_1 d_1 + \overline{b_1 d_1}$. If $\overline{b_2}d_2 = b_2\overline{d_2}$ then Im$(b_2\overline{d_2}) = 0$, and so $c^2 = \frac{\overline{b_2}d_2}{b_2\overline{d_2}} = \pm 1$, this is however a contradiction. So we only have one case to study, which is $\overline{b_2}d_2 + b_2\overline{d_2} = b_1 d_1 + \overline{b_1 d_1}$. A quick computation will show that Tr$(BD) = b_1 d_1 + \overline{b_1 d_1} - (\overline{b_2}d_2 + b_2\overline{d_2})$, and Tr$(DC^{-1}BC) =$ Re$(b_2\overline{d_2}c^2 - d_2\overline{b_2})$. This gives us that this fixed point set can be expressed as D such that D^{-1} is orthogonal to both B and $C^{-1}BC$ under the Frobenius inner product. Note that this is only true because Re$(b_2\overline{d_2}c^2 - d_2\overline{b_2}) = 0$ is equivalent to $b_2\overline{d_2}c^2 - d_2\overline{b_2} = 0$ because we are assuming $c^2 \neq 1$. So here we have our final two fixed point sets

$$U_4 = \{(\Gamma(0,a), B, \Gamma(c,0), \Gamma(\pm i\frac{|b_1|}{b_1}, 0)) | a, c \in S^1, B \in SU(2)\}$$
$$U_5 = \{(\Gamma(0,a), B, \Gamma(c,0), D) | a, c \in S^1, B, D \in SU(2), \langle B, D^{-1}\rangle_F = \langle C^{-1}BC, D^{-1}\rangle_F = 0\}$$

Here $\langle\cdot,\cdot\rangle_F$ is the Frobenius inner product. This gives us all of our fixed point sets, as this collection of cases is exhaustive. We can now show that the union of these fixed point sets is connected.

First consider U_1. If $[A,B,C,D]$ has a representative in U_1 then there exists some representative that diagonalizes both B and C, so $B = \Gamma(e^{i\theta_1}, 0)$ and $C = \Gamma(e^{i\theta_2}, 0)$. We can then define $b(t) = e^{i\theta_1(1-t)}$ and $c(t) = e^{i\theta_2(1-t)}$, and let $B(t), C(t)$ be obviously defined. As SU(2) is connected we know that there is some path $A(t), D(t)$ which connects each one to the identity. So taking all of these parts we have found $p(t) = (A(t), B(t), C(t), D(t))$ such that $p(0) = (A,B,C,D)$ and $p(1) = (Id, Id, Id, Id)$. As $[A,B,C,D]$ was chosen arbitrarily this proves that U_1 is connected.

Next consider U_2. If $[A, B, C, D]$ has a representative in U_2, then we can choose some representative where $C = \Gamma(i, 0)$ and $B = \Gamma(0, e^{i_\theta})$. We can define $b(t) = e^{i\theta(1-t)}$, $A(t), D(t)$ as before. Then we have $p(t) = (A(t), B(t), C(t), D(t))$ such that $p(0) = (A, B, C, D)$ and $p(1) = (Id, \Gamma(0, 1), \Gamma(i, 0), Id)$, and so as $[A, B, C, D]$ was chosen arbitrarily we have that U_2 is connected.

Now consider $U_1 \cup U_2 \cup U_3$. Given $(A, B, C, D) \in U_3$ we will show that there is a path to an element in U_1, and a path to an element in U_2. By the definition of U_3 we can assume $C = \Gamma(c, 0)$ with $c^4 \neq 1$. Let $c = e^{i\theta}$, and define $c(t) = e^{i(\theta+2\pi t)}$. This is a completely valid path in $U_1 \cup U_2 \cup U_3$, and there must be $t_1, t_2, t_3, t_4 \in (0, 1)$ such that $c(t_1) = 1$, $c(t_2) = -1$, $c(t_3) = i$, and $c(t_4) = -i$. Notice that if we define $p(t) = (A, B, C(t), D)$ then we have that $p(t_1), p(t_2) \in U_1$ and $p(t_3), p(t_4) \in U_3$. So because we have already shown that U_1 and U_2 are connected, we now have demonstrated the existence of a path from any element in U_3 to any element in U_1 or U_2. So we have the union is connected.

Now we can consider $\bigcup_{i=1}^{5} U_i$. Given any tuple $(A, B, C, D) \in U_4 \cup U_5$ we have that we can define $C(t)$ to be a path that sends C to Id and we have a path from an arbitrary element in U_4, U_5 to an element in U_1. If our element is in U_4 it is obvious how this works, as C is free. If our element lies in U_5 then continuity of the trace operator will give us that we can find a continuous $D(t)$ which will allow us to move C towards the identity while staying inside U_5.

This is all cases, and so we can conclude:

Theorem 4.3. *Let $\Phi_0, U_1, \ldots, U_5$ be as defined above. Then $[A, B, C, D] \in \chi(\Sigma, SU(2))$ is a fixed point of Φ_0 if and only if there exists a representative $(A, B, C, D) \in [A, B, C, D]$ such that*

$$(A, B, C, D) \in \bigcup_{i=1}^{5} U_i.$$

Moreover, this fixed point set is connected.

4.2 Curve with arbitrarily many loops

Let τ_k be the curve defined in Figure 5, viewed as a curve in Σ.

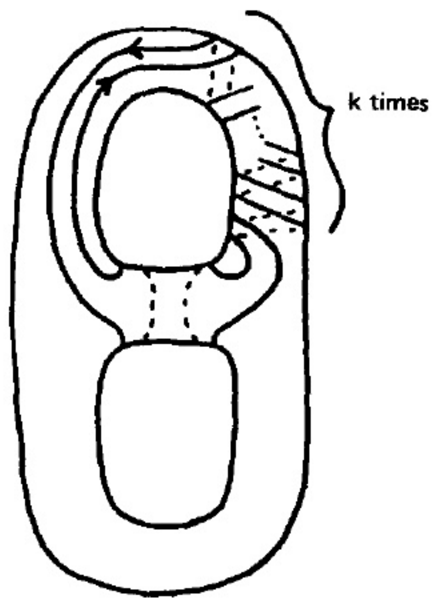

Figure 5: Illustrated here is a curve in $T^2\#T^2$, view this as a curve τ_k in Σ. Figure courtesy of [1, Fig. 4].

Define $\phi_k : \Sigma \to \Sigma$ to be the Dehn twist induced by τ_k. So ϕ_{k*} is the induced map on the fundamental group, and Φ_k is the induced map on the character variety, as in equation (1).

Theorem 4.4. *The fixed point set of Φ_k is homeomorphic to the fixed point set of Φ_0.*

Proof. Consider the generating set given by $\{\alpha, \beta_k := \beta\alpha^{-k}, \gamma, \delta\}$. We choose β_k this way as it allows us to write our curve as some cyclic permutation of $[\beta_k, \gamma]$ in a similar fashion to how we did for the $k = 0$ case. Using this we can see that the Dehn twist is defined by

$$\phi_{k*}(\alpha) = [\beta_k^{-1}, \gamma^{-1}]\alpha[\gamma, \beta_k^{-1}]$$

$$\phi_{k*}(\beta_k) = \beta_k$$

$$\phi_{k*}(\gamma) = \gamma$$

$$\phi_{k*}(\delta) = [\beta_k^{-1}, \gamma^{-1}]\delta[\gamma^{-1}, \beta_k]$$

which is exactly what we had above in the $k = 0$ case. So using this new generating set our problem has reduced entirely to the $k = 0$ case. □

5 References

References

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