Flat connections and holonomy

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Let $P \to X$ be a principal G-bundle over an oriented manifold X. By convention, G acts on P on the right. Denote by $\mathcal{A}(P)$ the set of connections on P, and $\mathcal{A}_{\text{flat}}(P)$ the set of flat connections. Let $\mathcal{G}(P)$ denote the gauge group. This is defined as the space of maps $u : P \to G$ that are equivariant:

$$u(p \cdot g) = g^{-1}u(p)g,$$

for all $p \in P$, $g \in G$. Then $\mathcal{G}(P)$ acts on $\mathcal{A}(P)$, and on $\mathcal{A}_{\text{flat}}(P)$. I will denote the action of $u \in \mathcal{G}(P)$ on $A \in \mathcal{A}(P)$ by u^*A.

In these notes I briefly sketch a proof of the well-known fact that

$$\sqcup_{[P]} \mathcal{A}_{\text{flat}}(P)/\mathcal{G}(P) \cong \text{hom}(\pi_1(X), G)/G, \tag{1}$$

where the action of G on the right-hand side is coming from conjugation, and the disjoint union is over the set of *equivalence classes* of bundles $P \to X$. See Kobayashi-Nomizu [4] for more details of the various assertions that follow.

1 Holonomy Maps

Fix a principal G-bundle $\pi : P \to X$. Given a connection $A \in \mathcal{A}(P)$ and a loop γ in X based at $x \in X$, the holonomy around γ is a G-equivariant map

$$\text{hol}_{A,x}(\gamma) : P_x \longrightarrow P_x,$$

where $P_x := \pi^{-1}(x)$ is the fiber over x. This is defined by parallel transport along γ. Fixing $p \in P_x$, we can interpret the holonomy as a Lie group element

$$\text{hol}_{A,x,p}(\gamma) \in G$$

by the identity

$$\text{hol}_{A,x}(\gamma)p = p \cdot \text{hol}_{A,x,p}(\gamma)$$

where $\cdot$ denotes the action of G on P. If A is a flat connection, then $\text{hol}_{A,x,p}(\gamma)$ depends on γ through its based homotopy class (essentially by definition of the term 'flat'), and so we have a map

$$\pi_1(X, x) \longrightarrow G, \qquad [\gamma] \longmapsto \text{hol}_{A,x,p}(\gamma) \tag{2}$$

2 A Group Homomorphism

Here we check that (2) defines a group homomorphism. It follows easily from the parallel transport definition of the holonomy that the map $P_x \to P_x$ is multiplicative:

$$\mathrm{hol}_{A,x}(\gamma_0 * \gamma_1) = \mathrm{hol}_{A,x}(\gamma_0) \circ \mathrm{hol}_{A,x}(\gamma_1),$$

where $*$ is concatenation of paths, and $\circ$ is composition of maps. Then the associated element of G satisfies

$$\mathrm{hol}_{A,x,p}(\gamma_0 * \gamma_1) = \mathrm{hol}_{A,x,p}(\gamma_0)\mathrm{hol}_{A,x,p}(\gamma_1),$$

where the concatenation on the right is the multiplication in G. When A is flat, the induced map (2) on $\pi_1(X, x)$ is obviously then a group homomorphism.

3 Dependence on Choices

The uniqueness of parallel transport implies that the map $\mathrm{hol}_{A,x,p}$ is equivariant in the sense that

$$\mathrm{hol}_{u^*A,x,p}(\gamma) = u(p)^{-1}\mathrm{hol}_{A,x,p}(\gamma)u(p),$$

for all gauge group elements $u \in \mathcal{G}(P)$. The holonomy therefore determines a map

$$\mathrm{hol}_{x,p} : \mathcal{A}_{\mathrm{flat}}(P)/\mathcal{G}(P) \longrightarrow \mathrm{hom}(\pi_1(X, x), G)/G,$$

where G acts on the right by conjugation by the inverse. Recall that different choices of x change $\pi_1(X, x)$ essentially by conjugation, so the space $\mathrm{hom}(\pi_1(X), G)/G := \mathrm{hom}(\pi_1(X, x), G)/G$ is independent of the choice of x. Similarly, the induced map $\mathrm{hol}_{x,p}$ is independent of the choices of $x \in X$ and $p \in P_x$.

4 Bijectivity of the Induced Map

Repeating the above for each equivalence class of principal G-bundle P gives a map

$$\mathrm{hol} : \sqcup_{[P]}\mathcal{A}_{\mathrm{flat}}(P)/\mathcal{G}(P) \longrightarrow \mathrm{hom}(\pi_1(X), G)/G.$$

(We need only union over the *equivalence classes*, and not all bundles, because we are working modulo $\mathcal{G}(P)$ and everything is equivariant.) A connection is determined by its holonomy, so it follows that this map hol is injective.

Now we show why every element of $\mathrm{hom}(\pi_1(X), G)/G$ can be realized as an element of $\mathcal{A}_{\mathrm{flat}}(P)/\mathcal{G}(P)$ for some principal G-bundle $P \to X$. Fix $\rho \in \mathrm{hom}(\pi_1(X, x), G)$, and let $\widetilde{X}$ be the universal cover of X. Then $\pi_1(X, x)$ acts on the bundle $\widetilde{X} \times G \to X$, where the action on the first factor is by deck transformations, and the action on the second factor is determined by ρ. Set

$$P := \left(\widetilde{X} \times G\right) / \pi_1(X, x).$$

This is naturally a principal G-bundle over X, and the trivial flat connection on $\widetilde{X} \times G$ descends to a connection A on P with the property that $\mathrm{hol}_{A,x,p} = \rho$.

5 Final Remarks

Here we only addressed bijectivity of the map (1). However, this map becomes an isomorphism in any reasonable category. For example, the space $\hom(\pi_1(X), G)/G$ has a natural topology coming from the topology on G. Endow $\mathcal{A}_{\text{flat}}(P)$ with, for example, the $\mathcal{C}^k$ topology, $\mathcal{G}(P)$ with the $\mathcal{C}^{k+1}$-topology, and $\mathcal{A}_{\text{flat}}(P)/\mathcal{G}(P)$ with the quotient topology. Then the holonomy is continuous with respect to these topologies, and one can check that the map (1) is a homeomorphism. Similar statements hold in the smooth category.

Finally, fix a principal bundle $P \to X$. One could ask how to determine, from a holonomy perspective, which elements of $\hom(\pi_1(X), G)/G$ are associated to P under the above construction. This is addressed in various settings in [1], [2, Section 4], and [3, Section 3.1].

References

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