Characterizing immutable sandpiles: A first look

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Abstract

By working with coefficients in $\mathbb{Z}$ or $\mathbb{R}$, one can define two different notions of stability for a sandpile on a graph. We call a sandpile *immutable* when these notions agree. Our main results give linear-algebraic characterizations for large classes of immutable sandpiles.

1 Introduction

Let Γ be a connected, finite multigraph without self-loops. Fix a vertex v_*, called the *sink*, and denote by $\mathcal{V}'$ the set of non-sink vertices of Γ. An *(abelian) sandpile* is any element of the set $\mathbb{Z}^{\mathcal{V}'}_{\geq 0}$ of nonnegative integer-labelings of $\mathcal{V}'$. Intuitively, one can view a sandpile as specifying a configuration of sand particles placed on the non-sink vertices. As we recall below, there is a notion of "stabilization" for sandpiles, which is an integral (discrete) process. Through the analogy with sand particles, stabilization can be viewed as a proxy for the beach-goer's observation of sand particles in a pile tumbling until a stable configuration is reached. On the other hand, via the inclusion $\mathbb{Z}^{\mathcal{V}'} \subseteq \mathbb{R}^{\mathcal{V}'}$, one can define a continuous analogue of stabilization for sandpiles. For any given sandpile, this continuous version of stabilization need not agree with the discrete version; when these notions do a agree, we call the sandpile "immutable". In this paper, we provide a characterization of the immutable sandpiles within a large class (Theorem 1.1).

Sandpiles were originally introduced by Bak–Tang–Wiesenfeld [1, 2], who showed that sandpiles exhibit a structure rich enough to model self-organizing criticality on lattices. This was later generalized to arbitrary graphs by Dhar [9], with further combinatorial ties elucidated by numerous researchers, including Björner–Lovász–Shor [5] and Biggs [4], who introduced the sink and much of the perspective we take here. The aforementioned rich structure of sandpiles arises through an equivalence relation called "legal toppling". To describe this, and to establish some notation, let $d : \mathcal{V} \to \mathbb{Z}$ be the degree (valance) map for the graph Γ, and let $\epsilon : \mathcal{V} \times \mathcal{V} \to \mathbb{Z}$ be the function specifying the number of edges joining two vertices. Here $\mathcal{V}$ is the vertex set for Γ. Given $\sigma \in \mathbb{Z}^{\mathcal{V}'}$, a vertex $v \in \mathcal{V}'$ is *unstable (for σ)* if $\sigma(v) \geq d(v)$. We say that σ is *unstable* if it has an unstable vertex; otherwise σ is called *stable*. A *toppling* of a vertex v is the new labeling $\sigma' \in \mathbb{Z}^{\mathcal{V}'}$ obtained from σ by reducing by $d(v)$ the value of $\sigma(v)$, and by increasing by $\epsilon(v, w)$ the value of $\sigma(w)$ at each vertex $w \neq v$. In the sand analogy, this is as if one vertex spills a grain of sand to each of its neighbors. If v is unstable for σ, then the toppling σ' is called *legal*. There is a unique stable sandpile τ_σ that is obtained from σ by a sequence of legal topplings [5]. We call τ_σ the *stabilization* of σ. The $\mathbb{Z}$*-odometer of* σ is the sandpile $u_\sigma^{\mathbb{Z}}$ defined so that $u_\sigma^{\mathbb{Z}}(v)$ is the number of times the vertex $v \in \mathcal{V}'$ is toppled during the stabilization of σ; that is, it measures the amount of sand released by each vertex in this stabilization process.

It was observed by Fey–Levine–Peres [10] that this has an equivalent least action-type characterization in which the $\mathbb{Z}$-odometer is realized as the unique minimizer of a certain system of matrix inequalities; we review this in Section 2. In this characterization, the minimization is taking place over the space $\mathbb{Z}^{\mathcal{V}'}$ of $\mathbb{Z}$-labelings of the non-sink vertices. In an effort to better understand scaling limits of sandpiles on integer lattices, Levine and Peres [15] considered the same system of matrix inequalities, but minimized over the larger space $\mathbb{R}^{\mathcal{V}'}$ of $\mathbb{R}$-labelings. In this larger system, there too is a unique minimizer $u_\sigma^{\mathbb{R}} \colon \mathcal{V}' \to \mathbb{R}$, which we call the $\mathbb{R}$*-odometer of* σ. The idea here is that if σ were allowed to topple through non-integral quantities, then it may stabilize to a configuration different than τ_σ. This can happen, and it turns out that the two stabilizations (integral and continuous) often exhibit very different qualitative features [15, 14].

There is a physical analogy for the continuous setting too: In place of the grains of sand, imagine a tower of a viscous liquid on each vertex, with mass-distribution described by σ. Suppose there were a vertex with too much fluid (more than the degree minus one). Then, over time, the fluid at this vertex would flow uniformly to its neighbors, until the vertex becomes stable in this new continuous sense. The $\mathbb{R}$-odometer measures the amount of fluid released by each vertex in this "$\mathbb{R}$-stabilization" process.

We will say a sandpile σ is *immutable* if $u_\sigma^{\mathbb{Z}} = u_\sigma^{\mathbb{R}}$; that is, if the two notions of stabilization for σ agree. Otherwise, we will call the sandpile *mutable*. The idea is that a mutable configuration is one for which a change of state (for example, from solid to liquid) can change its stabilization. To visualize this, the analogies above can be refined as follows: In place of sand particles, suppose now one has identical, homogenous metal spheres. Start with a sandpile consisting of such metal spheres, as well as one duplicate copy of that sandpile. Allow the first sandpile to stabilize in the usual sandpile sense. For the other sandpile, first increase the temperature to slightly beyond the melting point of the metal, and then allow it to stabilize. The initial sandpile is immutable if and only if the resulting stable configurations are identical.

The aim of the present paper is to better understand which sandpiles are immutable, and which are not. Our efforts were motivated by the following questions:

Q1. Do immutable sandpiles always exist? How about mutable sandpiles?

Q2. Is it more common for a sandpile to be mutable or immutable?

Q3. Is there a simple criterion for determining when a sandpile is immutable?

Q4. What does the set of all immutable sandpiles look like? Does it have any interesting structure?

We will see in Example 2.4 that every stable sandpile is immutable, so this provides an affirmative answer to the first question in Q1. For example, $\sigma = 0$ and $\sigma = d - 1$ are both stable and thus immutable; here and below we view the degree map d as a sandpile by restricting it to the non-sink vertices.

Our starting point for the remaining questions in Q1–Q4 is the following theorem, which provides sufficient conditions relative to which immutability is equivalent to the integrality of a certain vector quantity. To state it, let $L' \colon \mathbb{Z}^{\mathcal{V}'} \to \mathbb{Z}^{\mathcal{V}'}$ be the reduced Laplacian; depending on context, we may view this as a matrix over $\mathbb{Z}$ as indicated here, or over any subgroup of $\mathbb{R}$. Since Γ is connected, L' is invertible over $\mathbb{Q}$. We will say that $\sigma \in \mathbb{R}^{\mathcal{V}'}$ is *uniformly large* if $(L')^{-1}\sigma \geq (L')^{-1}(d-1)$, where the inequality is understood to be componentwise. The terminology is motivated by the following observation: If $\sigma \geq d-1$, then σ is automatically uniformly large, since the matrix entries of $(L')^{-1}$ are nonnegative [11], [3, Ch. 6].

Now we can state the central theorem of our paper.

Theorem 1.1. *If σ is a uniformly large sandpile, then the $\mathbb{R}$-odometer is*

$$u_\sigma^{\mathbb{R}} = (L')^{-1}(\sigma - d + 1).$$

In particular, σ is immutable if and only if $(L')^{-1}(\sigma - d + 1) \in \mathbb{Z}^{\mathcal{V}'}$ is integral.

The proof is given in Section 3.1. There, we also give examples in which the conclusion of the theorem does not hold in the absence of the hypothesis that σ is uniformly large.

With the characterization given in Theorem 1.1, we now begin to address Q1–Q4 above. For Q1, the following corollary establishes the existence of mutable sandpiles for a very large collection of multigraphs.

Corollary 1.2. *Assume there is a vertex v in Γ that is adjacent to v_* and satisfies the following:*

(a) if $w \neq v$, then w can be connected to v_* by an edge-path that does not contain v;

(b) v has degree $d(v) \geq 2$.

Then there is a sandpile on Γ that is mutable.

The proof, given in Section 3.3, uses the hypotheses on v to construct a uniformly large sandpile σ for which $(L')^{-1}(\sigma - d + 1)$ is not integral, at which point the result is immediate from Theorem 1.1. That said, the hypotheses of Corollary 1.2 are by no means fully necessary (see Example 4.3) and it seems it is rare for graphs to only admit immutable sandpiles. Nevertheless, these hypotheses cannot be altogether dropped either. For example, we will see in Section 4.1 and Example 4.2 that when $\Gamma = P_k$ is the path on $k = 2$ or $k = 3$ vertices, then there are no mutable sandpiles. These graphs fail the hypotheses of the corollary since every vertex on P_2 fails (b), while every vertex on P_3 fails either (a) or (b).

The questions in Q2–Q4 are more global in nature, and so are addressed by Theorem 1.1 only for the class of sandpiles that are uniformly large. Nevertheless, within this class of sandpiles, Theorem 1.1 is entirely satisfactory. Indeed, for Q3, the integrality of $(L')^{-1}(\sigma - d + 1)$ is a theoretically-pleasing condition for immutability, certainly when compared to the definition itself (in which one minimizes over solutions of a system of matrix inequalities). For Q4, the mapping $\sigma \mapsto (L')^{-1}(\sigma - d + 1)$ establishes an identification between the set of immutable, uniformly large sandpiles and the set $\mathbb{Z}_{\geq 0}^{\mathcal{V}'}$ of nonnegative integer labelings of $\mathcal{V}'$. We turn finally to Q2. This question is a heuristic one, so we give it a heuristic answer: Except for rare cases, the inverse reduced Laplacian $(L')^{-1}$ of a graph Γ is non-integral. Moreover, for a sandpile σ on any such Γ, the integrality of the quantity $(L')^{-1}(\sigma - d + 1)$ is generally much less likely than its non-integrality (by roughly a factor of the largest reduced denominator appearing in $(L')^{-1}$). That is, with the exception of a few special cases, it is much more likely for a uniformly large sandpile to be mutable than for it to be immutable.

Our next result restricts to a special class of graphs whose additional structure provides more refined information than that which is afforded by Theorem 1.1.

Corollary 1.3. *Suppose Γ is the cone of a regular graph, and choose the sink to be the cone point. Let $\sigma \in \mathbb{Z}_{\geq 0}^{\mathcal{V}'}$ be a sandpile.*

Then σ is immutable and uniformly large if and only if $\sigma = L'a$ for some $a \in \mathbb{Z}^{\mathcal{V}'}$ satisfying $a \geq d - 1$. When either (and hence both) of these conditions holds, the $\mathbb{R}$-odometer is $u_\sigma^{\mathbb{R}} = a - d + 1$.

See Section 3.2 for a proof. With Γ as in Corollary 1.3, this corollary makes an unexpected tie with the critical group $K(\Gamma)$ which, by definition, is the cokernel of the reduced Laplacian $L' \colon \mathbb{Z}^{\mathcal{V}'} \to \mathbb{Z}^{\mathcal{V}'}$. Indeed, if σ is uniformly large, then σ is immutable if and only if σ is a representative of the identity in $K(\Gamma)$.

The characterization of immutable sandpiles in Theorem 1.1 (and Corollary 1.3) essentially requires inverting the reduced Laplacian L'. Of course, the theoretical and computational tools available for this are vast; however, these strategies can be computationally taxing since inverting matrices is a highly nonlinear operation. It would be practically convenient to have a criterion for immutability that one could check directly, without having to invert a matrix. This would also be conceptually pleasing since, despite the elegance of the immutability criterion expressed in these results, precisely which class of sandpiles this identifies remains partially hidden behind the veil of inverting L'.

Towards this end, in Section 4, we give direct criteria for immutability of uniformly large sandpiles on dipoles, trees, complete graphs, and wheel graphs.

Through the results and examples mentioned above, we gain a fairly detailed picture of immutable and mutable sandpiles that are either uniformly large or stable (which can be viewed as "uniformly small"). Ideally, one would have a characterization of immutability in the absence of our uniformly large hypothesis (ULH) of Theorem 1.1. Section 4.1 and Example 4.2 give results in this direction, but their scope is very limited. As we have suggested above, throughout this paper, we probe the question of the necessity of our hypotheses through various examples that were particularly chosen to be simple, yet instructive. The moral of these examples is that the characterizations of immutability given in Theorem 1.1 and Corollary 1.3 would need to be significantly altered if the ULH were dropped. As such, the class of sandpiles that are neither uniformly large nor stable is not deeply explored here, and we leave for future work a more complete investigation of immutability for this class. We note also that it is precisely this class of sandpiles that produce the beautiful pictures in [15, 14]; indeed, these pictures are the stabilizations of mutable sandpiles that are point-masses on compactifications of $\mathbb{Z}^n$.

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2 The G-odometer

Let G be a subgroup of $(\mathbb{R}, +)$, and we assume $\mathbb{Z} \subseteq G$. The standard inequality on $\mathbb{R}$ induces a partial ordering $\leq$ on $G^{\mathcal{V}'}$. Given $\sigma \in \mathbb{Z}^{\mathcal{V}'} \subseteq G^{\mathcal{V}'}$, we will be interested in solving the following inequality system for u:

$$\sigma - L'u \leq d - 1 \tag{1}$$
$$u \geq 0 \tag{2}$$

The next proposition provides existence and uniqueness for minimizers of the system above.

Proposition 2.1 (Least Action Principle). *Assume that G is topologically closed as a subset of $\mathbb{R}$. For each $\sigma \in \mathbb{Z}^{\mathcal{V}'}$, there is a unique $u_\sigma^G \in G^{\mathcal{V}'}$ satisfying (1) and (2) that is minimal in the sense that $u_\sigma^G \leq u$ for all $u \in G^{\mathcal{V}'}$ satisfying (1) and (2).*

We prove this in Section 2.2 after we give several definitions, properties, and examples. The sandpile u_σ^G from the proposition will be called the G-

odometer of σ. Since $\mathbb{Z}^{\mathcal{V}'} \subseteq G^{\mathcal{V}'}$, we automatically have

$$u_\sigma^G \leq u_\sigma^{\mathbb{Z}}. \tag{3}$$

Consider the case $G = \mathbb{Z}$. The $\mathbb{Z}$-odometer $u_\sigma^{\mathbb{Z}}$ defined here is the same as the one discussed in the introduction; see [14, 10, 8]. Moreover, the stabilization τ_σ of σ from the introduction is given by $\tau_\sigma = \sigma - L' u_\sigma^{\mathbb{Z}}$.

We will say that σ is *G-immutable* if $u_\sigma^G = u_\sigma^{\mathbb{Z}}$. When $G = \mathbb{R}$ and there is little room for confusion, we will drop the $\mathbb{R}$ and simply say "immutable" in place of "$\mathbb{R}$-immutable". We will also say that a sandpile is *mutable* if it is not immutable.

Remark 2.2. In much of the sandpile literature (for example, [15, 13, 16]), the term "divisible sandpile" refers to real-valued maps $\sigma : \mathcal{V}' \to \mathbb{R}$. These objects are generally studied in a framework that is effectively separate from standard (integer-valued) sandpiles. This allows the freedom to consider the system

$$\sigma - L'u \quad \leq \quad 1 \tag{4}$$

for $u : \mathcal{V}' \to [0, \infty)$, which is a rescaled version of (1). However, our perspective in this paper is comparative in nature, in the sense that we want to simultaneously consider real-valued and integer-valued labelings. From our perspective, the rescaling leading to (4) is artificial, and so we consider (1) even for real-valued maps. ♢

2.1 Basic properties about G-immutable sandpiles

The following criterion follows directly from the definitions above, but is an observation we will use repeatedly.

Lemma 2.3. *A sandpile σ is G-immutable if and only if the G-odometer u_σ^G is an element of the subgroup $\mathbb{Z}^{\mathcal{V}'} \subseteq G^{\mathcal{V}'}$.*

Example 2.4. *(Every stable sandpile is G-immutable.)* If σ is stable then $\sigma \leq d - 1$, so $u = 0$ satisfies (1) and (2). This is clearly the minimal such solution over G, so $u_\sigma^G = 0$. The zero sandpile is integral, so $u_\sigma^{\mathbb{Z}} = u_\sigma^G$ by the previous lemma. ♢

Since $G \subseteq \mathbb{R}$, it follows that $u_\sigma^G \geq u_\sigma^{\mathbb{R}}$. This implies that σ is G-immutable whenever σ is $\mathbb{R}$-immutable. In fact, a partial converse of this holds: Assume that G contains all components of $(L')^{-1}$ (for example, $G \subseteq \det(L')^{-1}\mathbb{Z}$). Once we have Theorem 1.1 in hand, it will then follow that $u_\sigma^G = u_\sigma^{\mathbb{R}}$ for all uniformly large σ. Thus we have the following.

Corollary 2.5 (Corollary to Theorem 1.1)**.** *Assume L' is invertible over G and σ is uniformly large. Then σ is G-immutable if and only if σ is $\mathbb{R}$-immutable.*

The usefulness of this corollary is that, to determine immutability for uniformly large sandpiles, one can, for example, work in the cyclic group $\det(L')^{-1}\mathbb{Z}$ as opposed to $\mathbb{R}$.

2.2 Proof of Proposition 2.1

The proposition in the case $G = \mathbb{Z}$ is standard in the sandpile community; see [8, Ch. 6] for a proof in the case of finite multigraphs Γ, as considered here. In particular, there exists a unique integral minimizer $u_\sigma^{\mathbb{Z}} \in \mathbb{Z}^{\mathcal{V}'}$ of (1) and (2).

For general G, define $u_\sigma^G \colon \mathcal{V}' \to G$ by

$$u_\sigma^G(v) = \inf_u u(v),$$

where the infimum is over all $u \in G^{\mathcal{V}'}$ satisfying (1) and (2). Note that for each $v \in \mathcal{V}'$ this infimum is a well-defined element of G because (i) G is topologically closed in $\mathbb{R}$ and (ii) there does indeed exist an element of $G^{\mathcal{V}'}$ satisfying these inequalities: the $\mathbb{Z}$-odometer $u_\sigma^{\mathbb{Z}}$, whose existence we just established. We also have that u_σ^G satisfies (2), since inequalities are preserved by infima. We need to show that u_σ^G satisfies (1); once this has been done, it will follow immediately from the definition that u_σ^G is the *unique* minimal solution.

We begin with a preliminary computation: If u_1, u_2 satisfy (1) and (2), we will show that $\min(u_1, u_2)$ does as well. Let $u = \min(u_1, u_2)$ denote this minimum. Clearly u satisfies (2), so we turn to (1). To see this, fix a vertex $v \in \mathcal{V}'$. Without loss of generality, we may assume $u(v) = u_1(v) \le u_2(v)$. Using $\sim$ to denote adjacency, we have

$$\begin{aligned}
(L'u)(v) &= d(v)u(v) - \sum_{w\sim v} \epsilon(v,w)u(w) \\
&= d(v)u_1(v) - \sum_{w\sim v} \epsilon(v,w)u(w) \\
&\ge d(v)u_1(v) - \sum_{w\sim v} \epsilon(v,w)u_1(w) \\
&= (L'u_1)(v) \\
&\ge \sigma(v) - d(v) + 1.
\end{aligned}$$

Hence u satisfies (1).

With this in hand, for each vertex $v \in \mathcal{V}'$, fix a sequence $u_n^v \in G^{\mathcal{V}'}$ satisfying (1) and (2) with $u_n^v(v) \to u_\sigma^G(v)$. Define a new sequence u_n' by $u_n' = \min_v(u_n^v)$, where the minimum is taken over all $v \in \mathcal{V}'$. Clearly u_n' satisfies (2). We also have

$$|u_n'(v) - u_\sigma^G(v)| = u_n'(v) - u_\sigma^G(v) \le u_n^v(v) - u_\sigma^G(v) = |u_n^v(v) - u_\sigma^G(v)| \to 0$$

for all $v \in \mathcal{V}'$. This shows that u_n' converges to u_σ^G in the space $G^{\mathcal{V}'}$ of G-valued maps on $\mathcal{V}'$. Since matrix multiplication is continuous, this gives $L'u_n' \to L'u_\sigma^G$. The preliminary computation of the previous paragraph shows that u_n' satisfies (2), so for each $v \in \mathcal{V}'$ we have

$$\sigma(v) - d(v) + 1 \le (L'u_n')(v).$$

Taking the limit in n implies $\sigma(v) - d(v) + 1 \le (L'u_\sigma^G)(v)$, which is (1). □

2.3 Uniformly large sandpiles

Recall from the introduction that if $a \in \mathbb{R}^{\mathcal{V}'}$ is such that $a \geq d - 1$, then a is uniformly large (so $(L')^{-1}a \geq (L')^{-1}(d-1)$). The following example shows that the converse of this is false.

Example 2.6. *(A uniformly large sandpile σ where $\sigma \geq d - 1$ fails.)* Consider the case where $\Gamma = P_3$ is the path on three vertices, with sink v_* at one of the two endpoints. Order the remaining vertices v_1, v_2 linearly, with v_1 adjacent to v_*. This induces an identification $\mathbb{R}^{\mathcal{V}'} \cong \mathbb{R}^2$, and so each element $a \in \mathbb{R}^{\mathcal{V}'}$ can be viewed as a tuple $a = (a_1, a_2)^\dagger$, where the dagger denotes the transpose. Then the degree map restricted to $\mathcal{V}'$ is the vector $d = (2,1)^\dagger$, and the reduced Laplacian has inverse

$$(L')^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

Consider the tuple $\sigma = (0,1)^\dagger$. This is uniformly large since

$$(L')^{-1}(\sigma - d + 1) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

However, the inequality $\sigma \geq d - 1$ does not hold. $\diamondsuit$

Our last example in this section shows that there are elements of $\mathbb{R}^{\mathcal{V}'}$ that are uniformly large, but not sandpiles.

Example 2.7. *(A uniformly large sandpile σ with $L'\sigma$ not a sandpile.)* Let $\Gamma = K_3$ be the complete graph on three vertices. Label the non-sink vertices so $\mathbb{R}^{\mathcal{V}'} = \mathbb{R}^2$. The degree map on $\mathcal{V}'$ is $d = (2,2)^\dagger$, so clearly the vector $\sigma = (1,3)^\dagger$ satisfies $\sigma \geq d - 1$. To see this is uniformly large, a computation shows that $(L')^{-1}\sigma = (1, 5/6)^\dagger \geq d - 1$. Since $d - 1 = L'(d-1) = d - 1$ (Lemma 3.4), this implies that σ is uniformly large. However, $L'\sigma = (-1,5)^\dagger$ has a negative component, and so $L'\sigma$ is not a sandpile. $\diamondsuit$

3 Proofs of the main results

3.1 Proof and discussion of Theorem 1.1

We begin by proving Theorem 1.1 from the introduction. We then consider three examples that show the uniformly large hypothesis (ULH) cannot be removed entirely.

Proof of Theorem 1.1. Fix a sandpile σ. Since L' is invertible over $\mathbb{R}$, the equation

$$\sigma - L'\tilde{u} = d - 1 \tag{5}$$

has a unique solution $\tilde{u} \in \mathbb{R}^{\mathcal{V}'}$. When σ is uniformly large, we have $\tilde{u} = (L')^{-1}(\sigma - d + 1) \geq 0$. This and (5) imply that $\tilde{u} \in \mathbb{R}^{\mathcal{V}'}$ satisfies (1,2). In particular, since $u_\sigma^{\mathbb{R}}$ is minimal among all such solutions, we have $\tilde{u} - u_\sigma^{\mathbb{R}} \geq 0$. Set $q = \tilde{u} - u_\sigma^{\mathbb{R}}$. Note that the definition of $\tilde{u}$ combines with (1) for $u_\sigma^{\mathbb{R}}$ to give

$$L'q = \sigma - d + 1 - L'u_\sigma^{\mathbb{R}} \leq 0.$$

Thus $\langle L'q, q\rangle \leq 0$, since $q \geq 0$; the brackets denote the standard inner product on $\mathbb{R}^{\mathcal{V}'}$. On the other hand, L' is positive definite, so $0 \leq \langle L'q, q\rangle \leq 0$ and hence $q = 0$. This gives $u_\sigma^{\mathbb{R}} = (L')^{-1}(\sigma - d + 1)$. The remaining assertion of the theorem follows from Lemma 2.3. □

The following examples show that none of the claims of the theorem hold with the ULH dropped entirely. We begin with a very simple example.

Example 3.1. *(An immutable sandpile σ that is stable, not uniformly large, and for which all conclusions of Theorem 1.1 fail.)* Consider the sandpile $\sigma = 0$. This is stable, so $u_\sigma^{\mathbb{Z}} = u_\sigma^{\mathbb{R}} = 0$, which implies it is immutable. On the other hand, $(L')^{-1}(\sigma - d + 1) = (L')^{-1}(-d + 1)$. If Γ is any graph with $d \neq 1$, then $(L')^{-1}(-d + 1) \neq 0 = u_\sigma^{\mathbb{R}}$; this means the first conclusion of Theorem 1.1 would fail with the ULH dropped. If Γ is such that $(L')^{-1}(-d + 1)$ is not integral (such as the multigraph of Section 4.1 for $k \geq 2$), then this also means the second conclusion of the theorem would fail without the ULH. ♢

The preceding example is not entirely satisfying, since it still leaves room for the ULH to be replaced by something significantly simpler, such as "σ is not 0", or even "σ is not stable". The following example shows that neither of the conclusions of Theorem 1.1 would hold if the ULH were replaced by even the stronger of these two assumptions: that σ is not stable.

Example 3.2. *(An immutable sandpile σ that is unstable, not uniformly large, and for which all conclusions of Theorem 1.1 fail.)* Let $\Gamma = K_3$ be the complete graph on three vertices; we will use the vector notation of Example 2.7. Consider the sandpile $\sigma = (2, 0)^\dagger$. Note that this is not uniformly large, and it is not stable. One can compute directly that $(L')^{-1}(\sigma - d + 1) = (1/3, -1/3)^\dagger$. In particular, $(L')^{-1}(\sigma - d + 1)$ is not equal to $u_\sigma^{\mathbb{R}}$, since the $\mathbb{R}$-odometer is required to be nonnegative.

Next, we show that σ is mutable. Indeed, we will show

$$u_\sigma^{\mathbb{Z}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad u_\sigma^{\mathbb{R}} = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}.$$

For this, note that the system (1,2) is equivalent to

$$1 \leq 2x - y, \qquad 1 \geq x - 2y, \qquad x \geq 0, \qquad y \geq 0. \tag{6}$$

where we have written $u = (x, y)^\dagger$ in coordinates. From here, it is easy to see that the claimed values of $u_\sigma^{\mathbb{Z}}$ and $u_\sigma^{\mathbb{R}}$ satisfy (6) and hence (1, 2). It therefore suffices to show that these are minimal in $\mathbb{Z}$ and $\mathbb{R}$, respectively. For $u_\sigma^{\mathbb{Z}}$, minimality is easy since the only nonnegative vector that is smaller is $(0, 0)^\dagger$, and this does not satisfy (6). That $u_\sigma^{\mathbb{R}}$ is minimal follows from the first and last inequalities in (6), which imply $x \geq 1/2$. ♢

The preceding example still leaves open the possibility that the integrality of $(L')^{-1}(\sigma - d + 1)$ could detect immutability in the absence of the ULH. The following example shows this is not the case.

Example 3.3. *(A mutable sandpile σ that is unstable, not uniformly large, and for which all conclusions of Theorem 1.1 fail.)* Let $\Gamma = K_4$ be the complete graph on 4 vertices, and identify $\mathbb{R}^{\mathcal{V}'} \cong \mathbb{R}^3$ by ordering the vertices as in Example 2.7. Consider here the sandpile $\sigma = (4, 0, 0)^\dagger$. As in the previous example, one can check directly that

$$u_\sigma^{\mathbb{Z}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad u_\sigma^{\mathbb{R}} = \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix},$$

so σ is mutable. On the other hand,

$$(L')^{-1}(\sigma - d + 1) = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$$

which is integral. $\Diamond$

For similar phenomena, but on a path, see Example 4.3.

3.2 Proof and discussion of Corollary 1.3

Throughout this section, we assume that Γ is the cone of a regular graph and the sink is the cone point. We begin with two observations that are special to this setting; we then use these to prove Corollary 1.3. At the end, we discuss the extent to which the various hypotheses/conclusions of the corollary can be weakened/strengthened.

Lemma 3.4. *If Γ is the cone of a regular graph and the sink is the cone point, then*

$$d - 1 = L'(d - 1).$$

Proof. The assumption that Γ is a cone with sink given by the cone point implies that $L'c = c$ for any constant function $c \in \mathbb{R}^{\mathcal{V}'}$. Since Γ is the cone of a *regular* graph, it follows that d is constant on the non-sink vertices. Thus $d - 1$ is a constant sandpile, so the lemma follows. $\square$

Lemma 3.5. *Assume that Γ is the cone of a regular graph and the sink is the cone point. Fix a sandpile σ and write $\sigma = L'a$ for some $a \in \mathbb{R}^{\mathcal{V}'}$. If $a \geq d - 1$, then $u_\sigma^{\mathbb{R}} = a - d + 1$.*

Proof. The assumptions on a combine with Lemma 3.4 to give

$$\sigma - L'(a - (d - 1)) = d - 1, \quad a - (d - 1) \geq 0,$$

and so $u = a - (d-1)$ satisfies (1, 2). The minimality of $u_\sigma^{\mathbb{R}}$ then gives $u_\sigma^{\mathbb{R}} \leq a - (d-1)$. For the reverse inequality, notice that $\sigma + (d-1)$ is uniformly large since $\sigma \geq 0$ is a sandpile. Theorem 1.1 implies its $\mathbb{R}$-odometer is given by

$$u_{\sigma+(d-1)}^{\mathbb{R}} = (L')^{-1}(\sigma + (d-1) - (d-1)) = a.$$

By Lemma 3.4 again, we have

$$\sigma + (d-1) - L'(u_\sigma^{\mathbb{R}} + (d-1)) = \sigma - L'u_\sigma^{\mathbb{R}} \leq d-1,$$

Thus $u = u_\sigma^{\mathbb{R}} + (d-1)$ satisfies (1, 2) relative to the sandpile $\sigma + (d-1)$. By minimality of the $\mathbb{R}$-odometer for $\sigma + (d-1)$, we therefore have

$$a = u_{\sigma+(d-1)}^{\mathbb{R}} \leq u_\sigma^{\mathbb{R}} + (d-1),$$

which establishes the reverse inequality we were after. □

Proof of Corollary 1.3. First assume σ is a uniformly large sandpile. By Theorem 1.1, we have that σ is immutable if and only if $u_\sigma^{\mathbb{R}} = (L')^{-1}(\sigma - d + 1)$ is integral. Lemma 3.4 implies $(L')^{-1}(\sigma - d + 1) = (L')^{-1}(\sigma) - (d-1)$, so σ is immutable if and only if $a := (L')^{-1}\sigma$ is integral. This also shows that $a = u_\sigma^{\mathbb{R}} + (d-1) \geq d-1$, so a is uniformly large. This proves the "only if" direction.

For the converse, assume $\sigma = L'a \geq 0$ for some sandpile $a \geq d-1$. By Lemma 3.5, we have $u_\sigma^{\mathbb{R}} = a + d - 1$. Since a is integral, the immutability of σ is immediate from Lemma 2.3. □

We end this section with several examples illustrating that the various hypotheses in the statement of Corollary 1.3 cannot be dropped entirely. To begin, note that Example 2.7 shows we cannot simply drop the running hypothesis of Corollary 1.3 that σ is a sandpile; indeed, we cannot conclude that $\sigma = L'a$ is nonnegative only from the assumption that a is a uniformly large sandpile.

Next, we turn to the uniformly large hypothesis in the "only if" direction of Corollary 1.3. We give two examples showing that this hypothesis cannot be removed entirely. The first of these examples is a refinement of Example 3.1.

Example 3.6. *(An immutable sandpile σ that is not uniformly large, and is not in the image of L'.)* Let Γ be a graph with nontrivial critical group $K(\Gamma)$. Then there are stable (and thus immutable) sandpiles that are not in the image of $L' \colon \mathbb{Z}^{\mathcal{V}'} \to \mathbb{Z}^{\mathcal{V}'}$; by Corollary 1.3, these are not uniformly large.

To be concrete, consider $\Gamma = K_3$ with notation as in Example 2.7, and let $\sigma = (1,0)^\dagger$. In particular, σ is stable and thus is immutable. However, σ is not in the image of L' on $\mathbb{Z}^{\mathcal{V}'}$ since $(L')^{-1}\sigma = (2/3, 1/3)^\dagger \notin \mathbb{Z}^{\mathcal{V}'}$. ◊

We view stable sandpiles as "trivially immutable". As such, it would be more fulfilling to have an example similar to that of Example 3.6, but with σ unstable. This is supplied by the following.

Example 3.7. *(An immutable sandpile σ that is unstable, not uniformly large, and not in the image of L'.)* Let $\Gamma = K_3$ and consider the sandpile $\sigma = (4,0)^\dagger$. This is not uniformly large and not stable. As in Example 3.2, one can show that $u_\sigma^{\mathbb{Z}} = u_\sigma^{\mathbb{R}} = (2,1)^\dagger$ and so σ is immutable. However, σ is not in the image of $L' \colon \mathbb{Z}^{\mathcal{V}'} \to \mathbb{Z}^{\mathcal{V}'}$ since $(L')^{-1}\sigma = (8/3, 4/3)^\dagger$. $\lozenge$

Our last two examples illustrate that the hypothesis $a \geq d - 1$ in the "if" direction of Corollary 1.3 cannot be removed entirely.

Example 3.8. *(A mutable sandpile σ with $\sigma = L'a$ and where $a \geq d-1$ fails.)* Consider $\Gamma = K_4$ and $a = (2,1,1)^\dagger$. Note that the inequality $a \geq d-1$ does not hold. We have $L'a = (4,0,0)^\dagger \geq 0$, and so $\sigma = L'a$ is a sandpile. However, as we saw in Example 3.3, σ is mutable. $\lozenge$

Example 3.9. *(A non-uniformly large sandpile σ with $\sigma = L'a$ where $a \geq d-1$ fails.)* Consider $\Gamma = K_4$ and $a = (1,1,1)^\dagger$. Note that the inequality $a \geq d-1$ does not hold. We have $L'a = (1,1,1)^\dagger \geq 0$, and so $\sigma = L'a$ is a sandpile. However, σ is not uniformly large. $\lozenge$

3.3 Proof of Corollary 1.2

Let v be as in the statement of the corollary. Define a sandpile σ by $\sigma(v) = d(v)$ and $\sigma(w) = d(w) - 1$ for $w \neq v$. This is clearly uniformly large. By Theorem 1.1, to see that σ is mutable, it suffices to show that $(L')^{-1}(\sigma - d + 1) \in \mathbb{R}^{\mathcal{V}'}$ has a nonintegral component. To do this, we will show that the v-component of the vector $(L')^{-1}(\sigma - d + 1)$ lies in the interval $(0,1)$. By the cofactor formula for the inverse of L', it follows that the v-component of the vector $(L')^{-1}(\sigma - d + 1)$ is given by $L'_{v,v} / \det(L')$, where $L'_{v,v}$ is the minor of L' obtained by deleting the column and row corresponding to v. It therefore suffices to show that

$$0 < L'_{v,v} < \det(L'). \tag{7}$$

Our strategy is to use an extension of Kirchhoff's matrix tree theorem to reduce the problem to comparing counts of spanning trees and spanning 2-forests.

We will use the term 2-*forest* to refer to an unordered pair $\{\mathcal{T}_0, \mathcal{T}_1\}$ of pairwise disjoint subgraphs $\mathcal{T}_0, \mathcal{T}_1$ of Γ with the property that each $\mathcal{T}_i$ is a (connected) tree with at least one vertex. We will say a 2-forest is *spanning* if each vertex of Γ lies in one of the trees. (We caution the reader that our use of the term "spanning 2-forest" is a weaker notion than what is often meant by the term "spanning forest".) Any spanning tree determines a spanning 2-forest by deleting an edge. For $v, w \in \mathcal{V}'$, let $\mathcal{S}_2(v,w)$ be the set of ordered pairs $(\mathcal{T}_0, \mathcal{T}_1)$, where $\{\mathcal{T}_0, \mathcal{T}_1\}$ is a spanning 2-forest of Γ, $v_* \in \mathcal{T}_0$, and $v, w \in \mathcal{T}_1$. Let $\mathcal{S}_1$ be the set of all spanning trees of Γ.

Returning to the proof, it follows from Kirchhoff's matrix tree theorem that the determinant $\det(L') = |\mathcal{S}_1|$ is the number of spanning trees of Γ. Similarly, an extension [7] of the Matrix Tree Theorem shows that the minor $L'_{v,v} = |\mathcal{S}_2(v,v)|$ is the number of spanning 2-forests in Γ, with one tree containing v_* and the other containing v. We will show that (i) $\mathcal{S}_2(v,v)$ is not empty, and (ii) there is an injection $\mathcal{F} : \mathcal{S}_2(v, v) \hookrightarrow \mathcal{S}_1$ that is not surjective; these statements combine to give (7).

We will first show (i). Let $\mathcal{T}_1$ be the tree consisting of exactly the one vertex v. By the assumption (a) in the statement of the corollary, there is a disjoint tree $\mathcal{T}_0$ containing all vertices other than v, and so $(\mathcal{T}_0, \mathcal{T}_1) \in \mathcal{S}_2(v, v)$.

For (ii), recall we have assumed that v_* is adjacent to v. Fix one edge e that is incident to both v and v_*. Now let $(\mathcal{T}_0, \mathcal{T}_1) \in \mathcal{S}_2(v, v)$ and assume v_* is contained in $\mathcal{T}_0$. Note that neither $\mathcal{T}_0$ nor $\mathcal{T}_1$ contains e. Define $\mathcal{F}(\mathcal{T}_0, \mathcal{T}_1)$ to be the subgraph of Γ obtained from $\mathcal{T}_0 \cup \mathcal{T}_1$ by including this edge e. It is not hard to see that $\mathcal{F}(\mathcal{T}_0, \mathcal{T}_1)$ is a tree, and this is necessarily a spanning tree since all vertices of Γ are contained in it by construction. Thus we have a well-defined map of the form

$$\mathcal{F} : \mathcal{S}_2(v, v) \longrightarrow \mathcal{S}_1, \qquad (\mathcal{T}_0, \mathcal{T}_1) \longmapsto \mathcal{F}(\mathcal{T}_0, \mathcal{T}_1).$$

The injectivity of this map follows readily from the construction. Note that every spanning tree in the image of $\mathcal{F}$ contains the edge e. The failure of $\mathcal{F}$ to be surjective will therefore follow from the next claim.

Claim: There is a spanning tree of Γ that does not contain the edge e.

We have assumed in (b) in the statement of the corollary that the degree of v is at least 2. This implies there is another edge e' incident to v. If e' is also incident to v_*, then the claim follows by taking any spanning tree in the image of $\mathcal{F}$, deleting e and replacing it by e'. We may therefore assume that e' is incident to some other vertex $w \neq v, v_*$. Recall from (i) that there is a tree $\mathcal{T}$ in Γ that contains all vertices except v^*. In particular, $\mathcal{T}$ contains w and it does not contains e (nor any other edges adjacent to v), and so adding the edge e' to $\mathcal{T}$ produces the desired spanning tree. □

4 Examples

Here we explore immutability and Theorem 1.1 through the lens of four classes of graphs. We begin with dipoles, where we completely classify immutability in the absence of any uniformly large hypothesis. We then move on to trees, where we show that all uniformly large sandpiles are immutable. Though the analysis for these two classes of graphs is fairly simple, their conclusions are instructive. Lastly, we consider complete graphs and then wheel graphs. For these classes of graphs, we provide a direct classification of uniformly large immutable sandpiles in terms of systems of modular equations. The upshot from our perspective is that the conditions of these systems of equations can be checked directly from the sandpile, without having to invert a matrix.

4.1 Dipoles

Let Γ be a dipole; that is, a connected multigraph with two vertices, $k \geq 1$ edges, and no self-loops. See Figure 1. Fix a vertex to be the sink, and let v be the other vertex. We claim that a sandpile σ is immutable if and only if σ is stable or k divides $\sigma(v)+1$. (Note that $k = d(v)$ is the degree of v.)

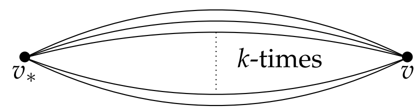

Figure 1: The dipole with 2 vertices, and k edges. $\Diamond$

By Example 2.4, it suffices to assume σ is unstable. Since there is only one non-sink vertex, the $\mathbb{R}$-odometer is the smallest real number $u(v) \geq 0$ so that

$$u(v) \geq (\sigma(v) + 1 - k)/k.$$

Clearly the minimum is $u_\sigma^{\mathbb{R}}(v) = (\sigma(v) + 1 - k)/k$, which is positive since σ is unstable. The claim now follows from this expression and Lemma 2.3.

If $1/k \in G$, this shows that G-immutability is equivalent to $\mathbb{R}$-immutability for any sandpile (Corollary 2.5 only applies to uniformly large sandpiles).

4.2 Trees

Assume that Γ is a tree and choose any vertex as the sink.

Theorem 4.1. *All uniformly large sandpiles are immutable.*

Proof. By Kirchhoff's matrix tree theorem, we have $\det(L') = 1$ and so L' is invertible over $\mathbb{Z}$. Thus $(L')^{-1}(\sigma - d + 1) \in \mathbb{Z}^{\mathcal{V}'}$ is integral for all $\sigma \in \mathbb{Z}^{\mathcal{V}'}$. If σ is uniformly large, then Theorem 1.1 implies $u_\sigma^{\mathbb{R}} = (L')^{-1}(\sigma - d + 1)$ and so σ is immutable. $\square$

Example 4.2. Let $\Gamma = P_3$ be a path with 3 vertices, labeled as in Example 2.6. We claim that all sandpiles on P_3 are immutable. By Theorem 4.1 and Example 2.4, it suffices to verify the claim for those sandpiles that are neither stable, nor uniformly large. The only sandpiles of this type are of the form $\sigma = (0, k)^\dagger$ for $k \geq 1$. In this case, the inequalities (1,2) are equivalent to

$$-2x + y \leq 1, \qquad k + x - y \leq 0, \qquad x, y \geq 0$$

for $u = (x, y)^\dagger$. Combining the first two, we have $k + x \leq y \leq 1 + 2x$, which implies $x \geq k - 1$, and so $y \geq 2k - 1$. It follows that $u_\sigma^{\mathbb{R}} = (k-1, 2k-1)^\dagger$ is the minimal such solution. This is integral, and so σ is immutable. $\Diamond$

Given the examples of Section 4.1 (with $k = 1$) and Example 4.2, one might suspect that all sandpiles on trees are immutable. The following example shows this is not the case.

Example 4.3. Consider the case where $\Gamma = P_4$ is the path with 4 vertices, labeled analogously to our labeling in Example 4.2. Thus

$$L' = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \qquad (L')^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}.$$

Consider the sandpile $\sigma = (2,0,0)^\dagger$. Arguing as in Example 4.2, one can show that $u_\sigma^{\mathbb{R}} = (1/2,0,0)^\dagger$, while $u_\sigma^{\mathbb{Z}} = (1,0,0)^\dagger$. In particular, σ is mutable.

We note also that $(L')^{-1}(\sigma - d + 1) = (0,-1,-1)^\dagger$ is integral and $u_\sigma^{\mathbb{R}} \neq (L')^{-1}(\sigma - d + 1)$. This therefore provides another example where the conclusions of Theorem 1.1 fail in the absence of the uniformly large hypothesis. $\Diamond$

4.3 Complete Graphs

Let $\Gamma = K_{n+1}$ be a complete graph on $n+1$ vertices and assume $n+1 \geq 3$. Fix any vertex to be the sink v_*. Label the remaining vertices $v_1, \ldots, v_n$ and thus identify $\mathbb{R}^{\mathcal{V}'} \cong \mathbb{R}^n$. For $\sigma \in \mathbb{R}^{\mathcal{V}'}$, write $(\sigma_1, \ldots, \sigma_n)$ for its components under this identification.

Theorem 4.4. *Suppose $\sigma \in \mathbb{Z}^{\mathcal{V}'}$. Then σ is in the image of $L' : \mathbb{Z}^{\mathcal{V}'} \to \mathbb{Z}^{\mathcal{V}'}$ if and only if*

$$\sigma_i - \sigma_j \equiv 0 \bmod n+1, \qquad \forall 1 \leq i, j \leq n. \tag{8}$$

In particular, if σ is uniformly large, then σ is immutable if and only if (8) holds.

Proof. We have

$$L' = \begin{pmatrix} n & -1 & \ldots & -1 \\ -1 & n & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & n \end{pmatrix}, \qquad (L')^{-1} = \frac{1}{n+1} \begin{pmatrix} 2 & 1 & \ldots & 1 \\ 1 & 2 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 2 \end{pmatrix}.$$

We will show that the image of L' is characterized by (8); the remaining claim of Theorem 4.4 will then be immediate from Corollary 1.3.

First assume σ satisfies (8). Thus, there exists $k \in \mathbb{Z}$ so that $\sigma_i = k \bmod n+1$ for all $1 \leq i \leq n$. Let $a = (L')^{-1}(\sigma) \in \mathbb{R}^{\mathcal{V}'}$ and write $a_i = a(v_i)$ for the ith component; we need to show that the a_i are integers. It follows from the matrix expression for $(L')^{-1}$ that

$$(n+1)a_i = 2\sigma_i + \sum_{j \neq i} \sigma_j = \sigma_i + \sum_j \sigma_j. \tag{9}$$

Temporarily working mod $n+1$, our hypothesis on σ gives

$$\sigma_i + \sum_j \sigma_j \equiv k + (kn) \equiv 0 \qquad \bmod n+1.$$

Thus $n+1$ divides $\sigma_i + \sum_j \sigma_j$ and so it follows from (9) that $a_i \in \mathbb{Z}$, as desired.

Conversely, assume that $\sigma = L'a$ for some $a \in \mathbb{Z}^{\mathcal{V}'}$. The formula for L' implies that

$$\sigma_i = na_i - \sum_{j \neq i} a_j = (n+1)a_i - \sum_j a_j.$$

Thus $\sigma_i - \sigma_j = (n+1)(a_i - a_j)$. Since the a_i are integers, this recovers (8). □

4.4 Wheel Graphs

Let $\Gamma = W_{n+1}$ be a wheel graph on $n+1$ vertices and assume $n+1 \geq 4$. Choose the sink v_* to be the central vertex, and label the boundary vertices $v_1, \dots, v_n$ cyclically. We treat the index of the v_i modulo n, so $i \in \mathbb{Z}/n\mathbb{Z}$; this reflects the rotational symmetry of W_{n+1}.

Given $\sigma \in \mathbb{R}^{\mathcal{V}'}$, we will write $\sigma_i = \sigma(v_i)$. Write F_k for the kth Fibonacci number, and A_k for the kth Lucas number. Our main result is as follows.

Theorem 4.5. *Let $\sigma \in \mathbb{Z}^{\mathcal{V}'}$. Then σ is in the image of $L' \colon \mathbb{Z}^{\mathcal{V}'} \to \mathbb{Z}^{\mathcal{V}'}$ if and only if the following holds for all $i \in \mathbb{Z}/n\mathbb{Z}$:*

$$n \text{ even:} \quad A_n \sigma_{i+n/2} + A_0 \sigma_i + \textstyle\sum_{m=1}^{\frac{n}{2}-1} A_{2m}(\sigma_{i+m} + \sigma_{i-m}) \equiv 0 \mod 5F_n \tag{10}$$

$$n \text{ odd:} \quad F_n \sigma_{i+(n+1)/2} + \textstyle\sum_{m=1}^{\frac{n-1}{2}} F_{2m-1}(\sigma_{i+m} + \sigma_{i+1-m}) \equiv 0 \mod A_n. \tag{11}$$

In particular, if σ is uniformly large, then σ is immutable if and only if (10,11) holds for all $i \in \mathbb{Z}/n\mathbb{Z}$.

The proof of Theorem 4.5 is given in Section 4.4.2. As with complete graphs, our proof relies on an explicit formula for the inverse of the reduced Laplacian, which is computed in Section 4.4.1.

The system (10) (resp. (11)) consists of n equations in the ring $\mathbb{Z}/5F_n\mathbb{Z}$ (resp. $\mathbb{Z}/A_n\mathbb{Z}$). Whenever one has a relatively prime factorization of $5F_n$ (resp. A_n), then the system (10) (resp. (11)) reduces in size. For example, if n is even and F_n and 5 are relatively prime, then the system (10) is equivalent to the same system of congruences, but in the smaller rings $\mathbb{Z}/5\mathbb{Z}$ and $\mathbb{Z}/F_n\mathbb{Z}$, simultaneously.

As another example, when $n = 2^k$ is a power of 2, we have $5F_{2^k} = 5\prod_{\ell=1}^{k-1} A_{2^\ell}$ (which can be seen by inducting on the identity $F_{2a} = A_a F_a$). This is a relatively prime factorization by [17]. As we show in Section 4.4.2, an extension of the argument of the previous paragraph produces the following.

Corollary 4.6. *Assume $n = 2^k$ for $k \geq 2$ and fix $\sigma \in \mathbb{Z}^{\mathcal{V}'}$. Then (10) holds for all $i \in \mathbb{Z}/n\mathbb{Z}$ if and only if*

$$\sum_{m=0}^{2^k-1} (-1)^m \sigma_m \equiv 0 \mod 5 \tag{12}$$

and, for all $1 \le \ell \le k-1$ *and* $0 \le i \le 2^\ell - 1$*,*

$$\sum_{c=0}^{2^{k-\ell}-1} (-1)^c \Big[A_0 \sigma_{i+2^\ell c} + \sum_{j=1}^{2^{\ell-1}-1} A_{2j} (\sigma_{i+2^\ell c+j} + \sigma_{i+2^\ell c-j}) \Big] \equiv 0 \mod A_{2^\ell}. \quad (13)$$

As suggested above, the main utility of Corollary 4.6 over Theorem 4.5 is that the rings of the former are significantly smaller. We note also the restriction on the index i in the corollary, which implies the systems expressed in (12) and (13) contain a total of $n-1 = 2^k - 1$ equations (so there is one less equation than (10)). Another simplification expressed by the corollary is that the Lucas numbers appearing in (12) and (13) are reduced mod 5 and A_{2^ℓ}, respectively. This reflects various Lucas number identities that are special for powers of 2; see Lemma 4.8. We encourage the reader to write out the system (10), and then (12,13) for $n = 2$ and 4 to get a sense for the symmetries expressed by the latter system that are hidden in the former.

Finally, we want to emphasize the helpfulness of experimental math, which we found to be an invaluable tool for identifying the results of this section. Indeed, the viability of the present wheel graph example began through various computations of immutability with the assistance of the computer algebra system Sage. When working with general (even) n, we initially observed no discernible pattern. However, when we specialized to $n = 2^k$, a pattern clearly emerged and we were able to conjecture the formulas of (12,13). After that, we wrote down a proof of the expression for the Laplacian given Theorem 4.7. We were once again aided by the computer, which suggested our initial formulas could be reduced significantly. Moreover, due to the overwhelming computational evidence, this led us to guess at the relative primality of A_{2^ℓ} for distinct $\ell \ge 1$. We are grateful to a Mathematics Stack Exchange answer by Peter Woolfitt for pointing out the reference [17], where this relative primality is established.

4.4.1 Counting trees and 2-forests

The main result of this section is the following explicit formula for the inverse of L'.

Theorem 4.7. *If n is even, then*

$$(L')^{-1} = \frac{1}{5F_n} \begin{pmatrix} A_n & A_{n-2} & A_{n-4} & \cdots & A_2 & A_0 & A_2 & \cdots & A_{n-2} \\ A_{n-2} & A_n & A_{n-2} & \cdots & A_4 & A_2 & A_0 & \cdots & A_{n-4} \\ A_{n-4} & A_{n-2} & A_n & \cdots & A_6 & A_4 & A_2 & \cdots & A_{n-6} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_2 & A_4 & A_6 & \cdots & A_n & A_{n-2} & A_{n-4} & \cdots & A_0 \\ A_0 & A_2 & A_4 & \cdots & A_{n-2} & A_n & A_{n-2} & \cdots & A_2 \\ A_2 & A_0 & A_2 & \cdots & A_{n-4} & A_{n-2} & A_n & \cdots & A_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n-2} & A_{n-4} & A_{n-6} & \cdots & A_0 & A_2 & A_4 & \cdots & A_n \end{pmatrix}.$$

If n is odd, then

$$(L')^{-1} = \frac{1}{A_n} \begin{pmatrix} F_n & F_{n-2} & F_{n-4} & \dots & F_3 & F_1 & F_1 & \dots & F_{n-2} \\ F_{n-2} & F_n & F_{n-2} & \dots & F_5 & F_3 & F_1 & \dots & F_{n-4} \\ F_{n-4} & F_{n-2} & F_n & \dots & F_7 & F_5 & F_3 & \dots & F_{n-6} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ F_3 & F_5 & F_7 & \dots & F_n & F_{n-2} & F_{n-4} & \dots & F_1 \\ F_1 & F_3 & F_5 & \dots & F_{n-2} & F_n & F_{n-2} & \dots & F_1 \\ F_1 & F_1 & F_3 & \dots & F_{n-4} & F_{n-2} & F_n & \dots & F_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ F_{n-2} & F_{n-4} & F_{n-6} & \dots & F_1 & F_1 & F_3 & \dots & F_n \end{pmatrix}$$

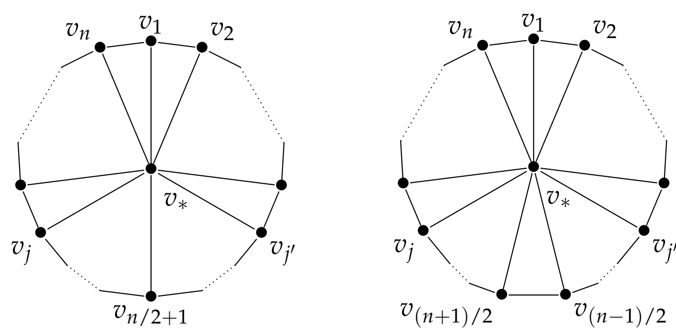

Figure 2: Illustrated here is the case where $j > n/2+1$. The figure on the left (resp. right) is for n even (resp. odd). Notice that in both pictures, reflecting across the vertical line from v_* to v_1 sends v_j to $v_{j'}$, where $j' = n+2-j$ (so $j' < n/2+1$). $\Diamond$

Proof. As shown in [6], the number of spanning trees of W_{n+1} is $\det(L') = A_{2n} - 2$. Using the notation and argumentation of Section 3.3, it follows that the (i, j)-component of the inverse of L' is given by

$$(L')^{-1}_{ij} = \frac{1}{A_{2n} - 2} |\mathcal{S}_2(v_i, v_j)|.$$

Due to the cyclic symmetry of wheel graphs, it suffices to compute this under the assumption that $i = 1$ and $1 \leq j \leq n$. (In the matrix for $(L')^{-1}$ claimed in the statement of Theorem 4.7, this symmetry manifests itself as each row being a cyclic permutation of the previous.) Similarly, we may assume that the indexing is such that, as we proceed from v_1 to v_j, we have gone no more than half-way around the boundary cycle; that is, we may assume

$$j \leq n/2 + 1. \tag{14}$$

This can be achieved by either relabeling the vertices in the opposite direction, or reflecting over the line through v_* and v_1, which amounts to the graph automorphism $v_i \mapsto v_{n+2-i}$. See Figure 2. (In the claimed matrix for $(L')^{-1}$, this reflection-symmetry manifests itself as the symmetry in the top row about the term A_0 when n is even, and between the two F_1 terms when n is odd.)

Fix $(\mathcal{T}_0, \mathcal{T}_1) \in \mathcal{S}_2(v_1, v_j)$. Since $v_* \in \mathcal{T}_0$, it follows that $\mathcal{T}_1$ will always be a path along the boundary cycle, containing v_1, v_j. In particular, the tree $\mathcal{T}_1$ contains at least j vertices (by (14)), and at most n vertices. For $j \leq m \leq n$, write $\mathcal{S}_{2,j}^m$ for the set of $(\mathcal{T}_0, \mathcal{T}_1) \in \mathcal{S}_2(v_1, v_j)$ with the property that $\mathcal{T}_1$ contains exactly m vertices. It therefore suffices to determine the size of each $\mathcal{S}_{2,j}^m$, since

$$|\mathcal{S}_2(v_1, v_j)| = \sum_{m=j}^{n} |\mathcal{S}_{2,j}^m|.$$

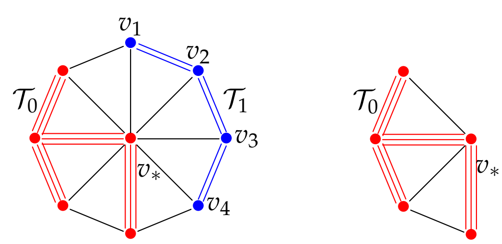

Figure 3: The figure on the left is the wheel graph W_9 on nine vertices. A spanning 2-forest $\{\mathcal{T}_0, \mathcal{T}_1\}$ is indicated. The tree $\mathcal{T}_1$ contains the vertices $v_1, \ldots, v_4$, with its edges indicated in red (tripled lines). The tree $\mathcal{T}_0$ contains the remaining vertices, with edges indicated in blue (doubled lines). The tuple $(\mathcal{T}_0, \mathcal{T}_1)$ can be viewed as an element of $\mathcal{S}_{2,j}^4$ for any $j \in \{1, \ldots, 4\}$. Note that $\mathcal{T}_1$ is a path consisting entirely of boundary vertices. On the right is the cone $C(P_4)$ obtained by deleting all vertices and edges associated to $\mathcal{T}_1$. Note that, given $\mathcal{T}_1$ and $j \in \{1, \ldots, 4\}$, the trees $\mathcal{T}_0$ for which $(\mathcal{T}_0, \mathcal{T}_1) \in \mathcal{S}_{2,j}^4$ are in one-to-one correspondence with the spanning trees of $C(P_4)$ on the right. $\Diamond$

Fix m with $j \leq m \leq n - (j-1)$. It follows that any $(\mathcal{T}_0, \mathcal{T}_1) \in \mathcal{S}_{2,j}^m$ must be such that $\mathcal{T}_1$ contains the smaller of the two arcs in the boundary k-cycle that connects v_1 and v_j. See Figure 3 for an example and Figure 4 for a non-example. There are $m - (j-1)$ choices of such $\mathcal{T}_1$, and we fix one $\mathcal{T}_1$. Remove from W_{n+1} all vertices of $\mathcal{T}_1$ as well as all edges adjacent to a vertex in $\mathcal{T}_1$. The result is the cone $C(P_{n-m})$ of a path P_{n-m} on $n - m$ vertices. Thus, any $\mathcal{T}_0$ with $(\mathcal{T}_0, \mathcal{T}_1) \in \mathcal{S}_{2,j}^m$ is a spanning tree for $C(P_{n-m})$. The number of spanning trees of $C(P_k)$ is F_{2k}; see [12]. Thus $|\mathcal{S}_{2,j}^m| = (m - (j-1))F_{2(n-m)}$.

When $n - (j-2) \leq m \leq n-1$, a boundary path $\mathcal{T}_1$ with m vertices can contain either of the two boundary arcs joining v_1 and v_j. See Figure 4. There are therefore $2m - n$ such paths $\mathcal{T}_1$. Fixing such a path $\mathcal{T}_1$, just as in the previous paragraph, there are $F_{2(n-m)}$ choices of $\mathcal{T}_0$ so that $(\mathcal{T}_0, \mathcal{T}_1) \in \mathcal{S}^m_{2,j}$. This gives $|\mathcal{S}^m_{2,j}| = (2m - n)F_{2(n-m)}$.

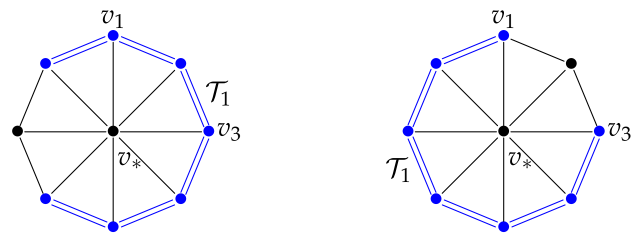

Figure 4: Each of the figures above illustrate an example of $\mathcal{T}_1$ on W_9 with $j = 3$ and $m = 7$; the vertices and edges of $\mathcal{T}_1$ are indicated in blue (doubled lines). The figure on the left contains the small arc in the boundary circle between v_1 and v_j. The figure on the right contains the large arc between v_1 and v_j. ♢

The last case to consider is when $m = n$. If $(\mathcal{T}_0, \mathcal{T}_1) \in \mathcal{S}^n_{2,j}$, then $\mathcal{T}_0$ necessarily consists only of v_*, and $\mathcal{T}_1$ can be any spanning tree of the boundary n-cycle obtained by deleting v_* from W_{n+1}. Thus, $|\mathcal{S}^n_{2,j}| = n$. In summary, we have

$$|\mathcal{S}_2(v_1, v_j)| = n + \sum_{m=j}^{n-(j-1)} (m - (j-1))F_{2(n-m)} + \sum_{m=n-(j-2)}^{n-1} (2m - n)F_{2(n-m)}$$

The aim now is to simplify this. At this point, it is convenient to change the summation index from m to $\ell = n - m$. Using the identities

$$\begin{aligned}
\sum_{\ell=a}^{b} \ell F_{2\ell} &= bF_{2b+1} - F_{2b} - aF_{2(a-1)+1} + F_{2(a-1)+2} \\
\sum_{\ell=a}^{b} F_{2\ell} &= F_{2b+1} - F_{2(a-1)+1}
\end{aligned}$$

we can write

$$\begin{aligned}
|\mathcal{S}_2(v_1, v_j)| &= n + \sum_{\ell=j-1}^{n-j} (n - \ell - j + 1)F_{2\ell} + \sum_{\ell=1}^{j-2} (n - 2\ell)F_{2\ell} \\
&= n + (n - j + 1) \sum_{\ell=j-1}^{n-j} F_{2\ell} - \sum_{\ell=j-1}^{n-j} \ell F_{2\ell} + n \sum_{\ell=1}^{j-2} F_{2\ell} - 2 \sum_{\ell=1}^{j-2} \ell F_{2\ell} \\
&= n + (n - j + 1)[F_{2(n-j)+1} - F_{2(j-2)+1}] \\
&\quad -[(n - j)F_{2(n-j)+1} - F_{2(n-j)} - (j - 1)F_{2(j-2)+1} + F_{2(j-2)+2}] \\
&\quad +n[F_{2(j-2)+1} - F_{2(0)+1}] \\
&\quad -2[(j - 2)F_{2(j-2)+1} - F_{2(j-2)} - (1)F_{2(0)+1} + F_{2(0)+2}] \\
&= F_{2(n-j)+1} + F_{2(n-j)} - F_{2(j-2)+2} + 2F_{2(j-2)+1} + 2F_{2(j-2)} \\
&= F_{2(n-j+1)} + F_{2(j-1)}
\end{aligned}$$

where, in the last line, we used the recursive relation for the Fibonacci numbers. The theorem is an immediate consequence of the following identity:

$$\textit{Claim: } \frac{1}{A_{2n}-2}\left(F_{2(n-j+1)}+F_{2(j-1)}\right) = \begin{cases} \frac{1}{5F_n}A_{n-2(j-1)} & \text{if } n \text{ is even} \\ \frac{1}{A_n}F_{n-2(j-1)} & \text{if } n \text{ is odd} \end{cases}$$

To prove this claim, recall the Fibonacci and Lucas numbers satisfy $F_{-a} = (-1)^{a+1}F_a$, $A_{-a} = (-1)^a A_a$, and

$$F_{a+b} = \tfrac{1}{2}(F_aA_b + A_aF_b) \qquad A_{a+b} = \tfrac{1}{2}(5F_aF_b + A_aA_b). \tag{15}$$

This implies

$$F_{a-b} = \frac{(-1)^b}{2}(F_aA_b - A_aF_b) \qquad A_{a-b} = \frac{(-1)^{b+1}}{2}(5F_aF_b - A_aA_b) \tag{16}$$

From which we obtain the identities $A_aF_b = F_{a+b} + (-1)^{b+1}F_{a-b}$ and $A_aA_b = A_{a+b} + (-1)^bA_{a-b}$. Freely referring to these identities, the following computation completes the proof of the claim when n is even:

$$\begin{aligned} (A_{2n}-2)A_{n-2(j-1)} &= (A_{2n}+2)A_{n-2(j-1)} - 4A_{n-2(j-1)} \\ &= A_n^2A_{n-2(j-1)} - 4A_{n-2(j-1)} \\ &= A_n(A_{2(n-j+1)} + A_{2(j-1)}) - 4A_{n-2(j-1)} \\ &= A_nA_{2(n-j+1)} + A_nA_{2(j-1)} - 4A_{n-2(j-1)} \\ &= 2A_{n-2(j-1)} + 5F_nF_{2(n-j+1)} \\ &\quad +2A_{n-2(j-1)} + 5F_nF_{2(j-1)} - 4A_{n-2(j-1)} \\ &= 5F_n(F_{2(n-j+1)} + F_{2(j-1)}). \end{aligned}$$

The case where n is odd is similar and left to the reader. □

4.4.2 Proofs of Theorem 4.5 and Corollary 4.6

Proof of Theorem 4.5. We prove the result under the assumption that n is even; the case where n is odd is similar. The wheel graph is the cone of a regular graph (a cycle graph), so by Corollary 1.3 it suffices to show that σ is in the image of L' if and only if (10) holds for all i. Define $M = 5F_n(L')^{-1}$; this is an integral matrix by Theorem 4.7. It follows that σ is in the image of $L'\colon \mathbb{Z}^{\mathcal{V}'} \to \mathbb{Z}^{\mathcal{V}'}$ if and only if $5F_n$ divides the ith component $\langle M\sigma, e_i\rangle$ of $M\sigma$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Using Theorem 4.7 again, we see that $\langle M\sigma, e_{i+n/2}\rangle$ is the left side of (10). This finishes the proof since $5F_n$ divides $\langle M\sigma, e_i\rangle$ for all $i \in \mathbb{Z}/n\mathbb{Z}$ if and only if $5F_n$ divides $\langle M\sigma, e_{i+n/2}\rangle$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. □

Our proof of Corollary 4.6 relies on the following equivalences for even Lucas numbers.

Lemma 4.8. *Let $j, \ell, c \in \mathbb{Z}$ with $\ell \geq 1$. Then*

$$\begin{array}{rcll} A_{2j} & \equiv & (-1)^j A_0 & \mod 5 \\ A_{2(j+c2^\ell)} & \equiv & (-1)^c A_{2j} & \mod A_{2^\ell}. \end{array}$$

Proof. Add (15) and (16) with $a = b = j$ to get $A_{2j} = 5F_j^2 + (-1)^j A_0$. The first identity of the lemma follows. For the second identity, use $A_{a+b} = A_a A_b - (-1)^b A_{a-b}$ with $a = 2j + 2^\ell$ and $b = 2^\ell$ to get

$$A_{2(j+2^\ell)} = A_{2j+2^\ell+2^\ell} = A_{2j+2^\ell} A_{2^\ell} - A_{2j} \equiv -A_{2j} \mod A_{2^\ell}.$$

This proves the identity for $c = 1$, and the identity for general $c \in \mathbb{Z}$ follows from induction. □

Proof of Corollary 4.6. Assume $n = 2^k$ for $k \geq 2$, fix $\sigma \in \mathbb{Z}^{\mathcal{V}'}$ and set

$$D_i = A_n \sigma_{i+2^{k-1}} + A_0 \sigma_i + \sum_{m=1}^{2^{k-1}-1} A_{2m}(\sigma_{i+m} + \sigma_{i-m}).$$

As discussed in the introduction to this section, it follows from Theorem 4.7 and the relatively prime decomposition $5F_{2^k} = 5\prod_{\ell=1}^{k-1} A_{2^\ell}$ that σ is immutable if and only if, for all $i \in \mathbb{Z}/2^k\mathbb{Z}$, the integer D_i is congruent to 0 mod 5, and congruent to 0 mod A_{2^ℓ} for all $1 \leq \ell \leq k-1$. Our aim is to simplify these conditions by exploiting the symmetries expressed in Lemma 4.8.

We begin with a preliminary computation. By the 2^k-periodicity of the index of σ_i, we have $\sigma_{i-m} = \sigma_{i+2^k-m}$, and so we can write

$$\begin{aligned} D_i &= \left(A_0\sigma_i + \sum_{m=1}^{2^{k-1}-1} A_{2m}\sigma_{i+m}\right) + \left(A_{2^k}\sigma_{i+2^k-2^{k-1}} + \sum_{m=1}^{2^{k-1}-1} A_{2m}\sigma_{i+2^k-m}\right) \\ &= \sum_{m=0}^{2^{k-1}-1} A_{2m}\sigma_{i+m} + \sum_{m=1}^{2^{k-1}} A_{2m}\sigma_{i+2^k-m} \\ &= \sum_{m=0}^{2^{k-1}-1} A_{2m}\sigma_{i+m} + \sum_{m=2^{k-1}}^{2^k-1} A_{2(2^k-m)}\sigma_{i+m} \end{aligned}$$

where we performed the change of index $m \mapsto 2^k - m$ in the second sum of the last line. It follows from Lemma 4.8 that $A_{2(2^k-m)} \equiv A_{-2m} = A_{2m}$, whenever working mod 5 or mod A_{2^ℓ}. Thus, we can continue the above to get

$$D_i \equiv \sum_{m=0}^{2^k-1} A_{2m}\sigma_{i+m} \quad \mod 5 \quad \text{or} \quad \mod A_{2^\ell}. \tag{17}$$

This is the desired preliminary computation.

Now consider the mod 5 case. Applying Lemma 4.8 again, we see immediately from (17) that

$$D_i \equiv A_0 \sum_{m=0}^{2^k-1} (-1)^m \sigma_{i+m} \qquad \mod 5.$$

Note first that this shows $D_{i+1} \equiv -D_i \mod 5$, and so this being congruent to 0 is independent of i. Since $A_0 = 2$ is invertible mod 5, it follows that $D_i \equiv 0 \mod 5$ for all i if and only if (12) holds (which corresponds to $i = 0$).

Next, fix $1 \leq \ell \leq k-1$ and work mod A_{2^ℓ}. By (17), we have

$$\begin{aligned} D_{i+2^\ell} &\equiv \sum_{m=0}^{2^k-1} A_{2m}\sigma_{i+m+2^\ell} \\ &\equiv \sum_{m=0}^{2^k-2^\ell-1} A_{2m}\sigma_{i+m+2^\ell} + \sum_{m=2^k-2^\ell}^{2^k-1} A_{2m}\sigma_{i+m+2^\ell} \\ &= \sum_{m=2^\ell}^{2^k-1} A_{2(m-2^\ell)}\sigma_{i+m} + \sum_{m=0}^{2^\ell-1} A_{2(m-2^\ell-2^k)}\sigma_{i+m} \end{aligned}$$

where we did a change of index in each sum appearing in the last line. By Lemma 4.8, we can continue this as

$$D_{i+2^\ell} \equiv -\sum_{m=2^\ell}^{2^k-1} A_{2m}\sigma_{i+m} - \sum_{m=0}^{2^\ell-1} A_{2m}\sigma_{i+m} = -D_i \qquad \mod A_{2^\ell}.$$

Thus $D_i \equiv 0 \mod A_{2^\ell}$ for all $i \in \{0, \ldots, 2^k - 1\}$ if and only if $D_i \equiv 0 \mod A_{2^\ell}$ for all $i \in \{0, \ldots, 2^\ell - 1\}$ (the latter being a considerably smaller system).

Finally, we need to compute D_i mod A_{2^ℓ} and show it has the claimed form. For this, return to the expression (17) and note that each $m \in \{0, \ldots, 2^k - 1\}$ can be expressed as $m = 2^\ell c + j$ for unique $c \in \{0, \ldots, 2^{k-\ell} - 1\}$ and $j \in \{0, \ldots, 2^\ell - 1\}$. Using Lemma 4.8 again, we have

$$\begin{aligned} D_i &\equiv \sum_{c=0}^{2^{k-\ell}-1} \sum_{j=0}^{2^\ell-1} A_{2(2^\ell c+j)}\sigma_{i+2^\ell c+j} \\ &\equiv \sum_{c=0}^{2^{k-\ell}-1} (-1)^c \sum_{j=0}^{2^\ell-1} A_{2j}\sigma_{i+2^\ell c+j} \end{aligned}$$

Note that when $j = 2^{\ell-1}$ we have $A_{2j} \equiv 0$. Thus, we can write

$$\begin{aligned} D_i \equiv &\sum_{c=0}^{2^{k-\ell}-1} (-1)^c \Big[A_0\sigma_{i+2^\ell c} \\ &+ \sum_{j=1}^{2^{\ell-1}-1} A_{2j}\sigma_{i+2^\ell c+j} + \sum_{j=2^\ell+1}^{2^\ell-1} A_{2j}\sigma_{i+2^\ell c+j} \Big] \\ \equiv &\sum_{c=0}^{2^{k-\ell}-1} (-1)^c \Big[A_0\sigma_{i+2^\ell c} \\ &+ \sum_{j=1}^{2^{\ell-1}-1} A_{2j}\sigma_{i+2^\ell c+j} + \sum_{j=1}^{2^{\ell-1}-1} A_{2(2^{\ell-1}+j)}\sigma_{i+2^\ell c+2^{\ell-1}-j} \Big] \end{aligned}$$

where we did an index change in the second sum. Use $A_{2(2^{\ell-1}+j)} \equiv -A_{2(2^{\ell-1}-j)}$ and another reindexing to continue this as

$$\begin{aligned} D_i \equiv & \sum_{c=0}^{2^{k-\ell}-1} (-1)^c \Big[A_0 \sigma_{i+2^\ell c} \\ & + \sum_{j=1}^{2^{\ell-1}-1} A_{2j} \sigma_{i+2^\ell c+j} + \sum_{j=1}^{2^{\ell-1}-1} (-1) A_{2j} \sigma_{i+2^\ell (c+1)-j} \Big] \end{aligned}$$

Focus on the last term (its sum over both c and j), and write this as

$$\begin{aligned} \sum_{c=0}^{2^{k-\ell}-1} (-1)^c \sum_{j=1}^{2^{\ell-1}-1} (-1) A_{2j} \sigma_{i+2^\ell(c+1)-j} &= \sum_{c=0}^{2^{k-\ell}-1} (-1)^{c+1} \sum_{j=1}^{2^{\ell-1}-1} A_{2j} \sigma_{i+2^\ell(c+1)-j} \\ &= \sum_{c=1}^{2^{k-\ell}} (-1)^c \sum_{j=1}^{2^{\ell-1}-1} A_{2j} \sigma_{i+2^\ell c-j} \\ &\equiv \sum_{c=0}^{2^{k-\ell}-1} (-1)^c \sum_{j=1}^{2^{\ell-1}-1} A_{2j} \sigma_{i+2^\ell c-j} \end{aligned}$$

where the last equality holds by the 2^k-periodicity of the index of σ_i, which implies the $c = 0$ term equals the $c = 2^\ell$ term. In summary, this gives

$$D_i \equiv \sum_{c=0}^{2^{k-\ell}-1} (-1)^c \Big[A_0 \sigma_{i+2^\ell c} + \sum_{j=1}^{2^{\ell-1}-1} A_{2j} (\sigma_{i+2^\ell c+j} + \sigma_{i+2^\ell c-j}) \Big].$$

□

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