Varieties, Orbit Spaces, and the Heisenberg Group H_3

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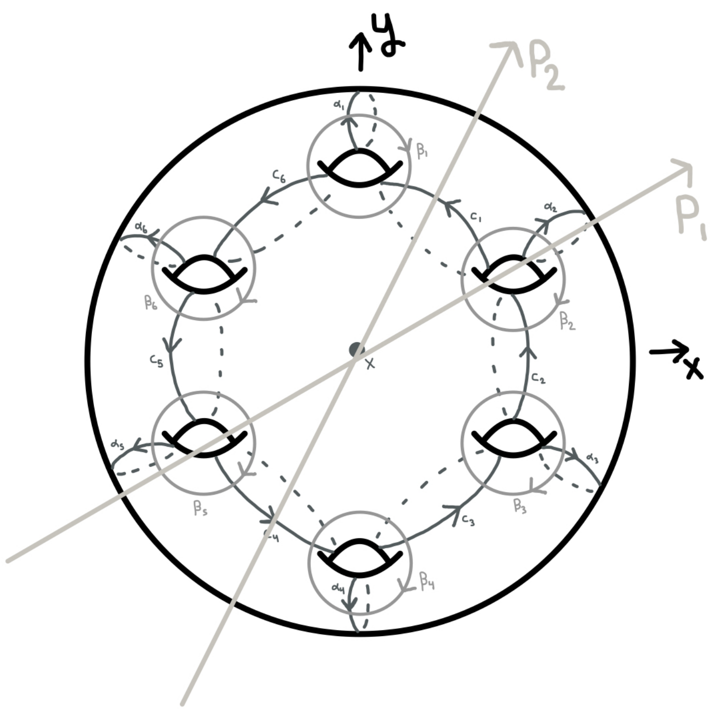


Introduction

This note arose out of a 6 week undergraduate research project with Dr. David Duncan, that consisted of an investigation related to generators of surface mapping class groups and induced fixed point sets of various homeomorphisms. The first half of this document consists of an undergraduate level introduction to Representation Varieties, Orbit Spaces, and the Heisenberg Group H_3. The second half provides an example of a potential application in Geometric Topology. The intention is to provide a reference document for future students looking to perform research in this area. The concepts we explore are shown to be particularly useful in the study of surface mapping class groups and fixed point sets of homeomorphisms. This is a fruitful area of Geometric Topology research at the undergraduate level. These notes are suitable for someone who has taken courses in Linear Algebra and introductory proof writing, and has a basic understanding of Group Theory and Algebraic Topology.

The Representation Variety

Let π be a finitely presented group with the generators $a_1, a_2, ..., a_n$, and G be any group. A homomorphism $f : \pi \to G$ is called a G-Representation of π. The set of all G-representations of π is called the *G-Representation Variety* of π. We denote this as $R(\pi, G)$.

Since π is finitely presented, every representation $f \in R(\pi, G)$ is determined by how it acts on each of $a_1, a_2, ..., a_n$. We can then denote any $f \in R(\pi, G)$ as:

$$\Big(f(a_1), f(a_2), ..., f(a_n)\Big) = (A_1, A_2, ..., A_n)$$

where each of $A_1, A_2, ..., A_n$ are elements of G. This gives us that $R(\pi, G)$ is a subspace of G^n, where n is the number of generators of π.

In Geometric Topology, a typical choice of π is the fundamental group of a closed, connected, oriented surface Σ_g of genus g. We denote this by $\pi_1(\Sigma_g)$. The generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ with the relation $\Pi_{i=1}^{g}[\alpha_i, \beta_i] = Id$ provide a finite presentation of $\pi_1(\Sigma_g)$. We call $\Pi_{i=1}^{g}[\alpha_i, \beta_i] = Id$ the product of commutators condition. Furthermore, the notation for the G-representation variety of $\pi_1(\Sigma_g)$ is often shortened to $R(\Sigma_g, G)$.

Orbit Space

An Orbit Space is built from $R(\Sigma_g, G)$ as follows:

$$\chi(\Sigma_g, G) = \frac{R(\Sigma_g, G)}{G}$$

where the action is conjugation; for $f, f' \in R(\Sigma_g, G)$, $f \sim f'$ if and only if there exists an $a \in G$ such that $f(x) = af'(x)a^{-1}$ for all $x \in \pi_1(\Sigma_g)$. Thus $\chi(\Sigma_g, G)$ is the set of conjugacy classes in $R(\Sigma_g, G)$. We will then denote $[f] \in \chi(\Sigma_g, G)$ as:

$$\Big[f(\alpha_1), f(\beta_1), ..., f(\alpha_g), f(\beta_g)\Big] = [A_1, B_1, ..., A_g, B_g]$$

The equivalence is given by simultaneous conjugation on each element.

Remark: In the literature, G is often taken to be a compact Lie group, giving the *character variety* $\chi(\Sigma_g, G)$. This is where the use of χ in our notation originates.

The Group H_3

The *Heisenberg group* H_3 consists of upper triangular matrices with real valued entries of the form:

$$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

with the group action being matrix multiplication. It is a nilpotent real Lie group of dimension 3, and non-compact. For ease of notation and computation, we will denote matrices of this form as $A = \aleph(a_A, b_A, c_A)$. In this group, multiplication of two elements $A = \aleph(a_A, b_A, c_A)$ and $B = \aleph(a_B, b_B, c_B)$ gives $AB = \aleph(a_A + a_B, b_A + b_B, c_A + a_Ab_B + c_B)$ which is not generally commutative. Inverses are given by $A^{-1} = \aleph(-a_A, -b_A, a_Ab_A - c_A)$. The center of the group consists of elements of the form A=$\aleph(0, 0, c_A)$. This group behaves particularly well during algebraic computation. The following lemma is an example:

Lemma 1. $[A, B] = \aleph(0, 0, a_Ab_B - a_Bb_A)$

Proof. Expand the left hand side as $ABA^{-1}B^{-1}$ and verify by direct computation. This computation is a useful exercise for the reader. □

Notice from the above computations that:

$$a_A b_B - a_B b_A = \det \begin{bmatrix} a_A & b_A \\ a_B & b_B \end{bmatrix}$$

This observation gives the following important property:

Proposition 1. *$[A, B] = Id$ if and only if $a_A b_B - a_B b_A = 0$.*

Proof. From the above, we have:

$$\det \begin{bmatrix} a_A & b_A \\ a_B & b_B \end{bmatrix} = 0 \Leftrightarrow [A, B] = \aleph(0, 0, 0) = Id$$

□

Another useful property of H_3 is shown in the following proposition. This proves useful in future computations:

Proposition 2. *For all $A \in H_3$, $A^n = Id$ if and only if $A = I$.*

Proof. Let $A^n = Id$, so that:

$$\aleph(na, nb, \_\_) = \aleph(0, 0, 0)$$

We then have that $a = 0$ and $b = 0$. This then says that A is in the center of H_3, and thus:

$$A^n = \aleph(0, 0, nc)$$

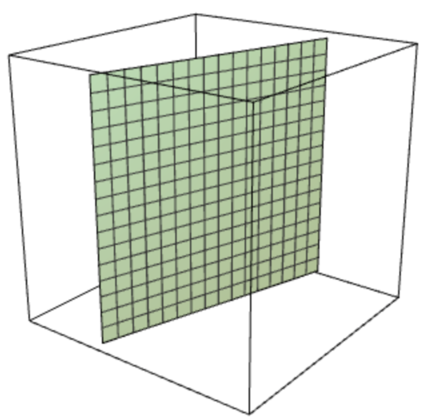
This gives that $\aleph(0, 0, nc) = \aleph(0, 0, 0)$, so $c = 0$ completing the proof. □

Note that commuting matrices in H_3 have nice general forms. Proposition 3 demonstrates this:

Proposition 3. *Two matrices of the form $A = \aleph(ma, mb, c_1)$ and $B = \aleph(na, nb, c_2)$ commute.*

Proof. Compute AB and BA. It is a useful exercise for the reader to verify that $AB = BA$. □

We will now take a geometric view of centralizers in H_3. Recall that for $A \in H_3$, $C_A = \{B \mid [A, B] = Id\}$. If $A = \aleph(a_A, b_A, c_A)$ and $B = \aleph(a_B, b_B, c_B)$, then we need $a_A b_B - a_B b_A = 0$. Since a_A and b_A are fixed based on A, this means a_B and b_B lie on a line in $\mathbb{R}^3$. But c_B can be any real number and B will still satisfy the above condition, so this line extends to a plane in $\mathbb{R}^3$ containing the c_A axis, with two of B's components lying on the line determined by a_A and b_A. The figure below shows what an arbitrary space of centralizers might look like.



Furthermore, we have the following lemma which will again prove useful in future computations:

Lemma 2. *If $B^2 \in C_A$, then $B \in C_A$.*

Proof. $B^2 = \aleph(2a_B, 2b_B, c)$ and since $B^2 \in C_A$, we have $a_A(2b_B) = (2a_B)b_A \Leftrightarrow a_A b_B = a_B b_A$. Proposition 1 then gives that $[A, B] = Id$, and thus $B \in C_A$ as required. □

As motivation for the use of $G = H_3$ in the study of representation varieties, orbit spaces, and their applications, we note that there are many nice topological properties of the spaces H_3, $R(\Sigma_g, H_3)$ and $\chi(\Sigma_g, H_3)$. Furthermore, H_3 is nearly Abelian. This causes many computations to simplify greatly where other choices of G would lead to complication. As examples of properties we gain by choosing $G = H_3$, take the following theorems:

Theorem 1. *The space H_3 is contractible.*

Proof. Let $X = \aleph(x_1, x_2, x_3)$ be arbitrary in H_3. We construct a function $\varphi : H_3 \times [0, 1] \to H_3$. Define:

$$\varphi(X, t) = \aleph(tx_1, tx_2, tx_3)$$

We then have $\varphi(X, 0) = \aleph(0, 0, 0) = Id$, and $\varphi(X, 1) = \aleph(x_1, x_2, x_3) = X$. The function φ is continuous on $H_3 \times [0, 1]$. Thus H_3 is contractible. □

Theorem 2. *The space $R(\Sigma_g, H_3)$ is contractible.*

Proof. Let $f = (A_1, B_1, A_2, B_2, ..., A_g, B_g) \in R(\Sigma_g, H_3)$. From the product of commutator condition on $\pi_1(\Sigma_g)$, we have $\Pi_{i=1}^{g}[A_i, B_i] = Id$. From each pair $[A_i, B_i]$ we get a matrix of the form $\aleph(0, 0, a_{A_i}b_{B_i} - a_{B_i}b_{A_i})$. Denote $a_{A_i}b_{B_i} - a_{B_i}b_{A_i}$ as c_i. Then the product of commutators condition expands to $\aleph(0, 0, \Sigma_{i=1}^{g}c_i) = \aleph(0, 0, 0)$ and thus we have $\Sigma_{i=1}^{g}c_i = 0$. We construct a function $\varphi : R(\Sigma_g, H_3) \times [0, 1] \to R(\Sigma_g, H_3)$. Define:

$$\varphi(f, t) = \Big(A_1(t), B_1(t), ..., A_g(t), B_g(t)\Big)$$

where $X(t) = \aleph(ta_M, tb_M, tc_M)$. This gives $\varphi(f, 0) = (Id, Id, ..., Id, Id)$. Similarly, $\varphi(f, 1) = (A_1, B_1, ..., A_g, B_g)$. We must check that:

$$\prod_{i=1}^{g}[A_i(t), B_i(t)] = Id$$

From the above it can be verified computationally that this gives $\Sigma_{i=1}^{g}t^2c_i = 0$, but we can factor out t^2 giving $\Sigma_{i=1}^{g}c_i = 0$, which is a tautology. This gives that φ is continuous on $R(\Sigma_g, H_3) \times [0, 1]$. Thus $R(\Sigma_g, H_3)$ is contractible. The reader is encouraged to fill in the computational steps of this proof. □

Motivation for the Study of Representation Varieties and Orbit Spaces

A fundamental question in Topology is understanding how a surface homeomorphism acts on objects such as the fundamental group. Expressing a homeomorphism algebraically provides insight into this. However, the fundamental group is often difficult to work with computationally, so studying representations can be a useful approach. A homeomorphism ϕ induces a pullback ϕ^* on both the representation variety and orbit space. This means we can understand properties of ϕ by looking at actions of the pullback on these objects. Furthermore, representation varieties and orbit spaces arise as sets of representations, and are topological spaces in their own right. This allows for topological and geometric analysis of these spaces.

An Application with Mapping Class Groups

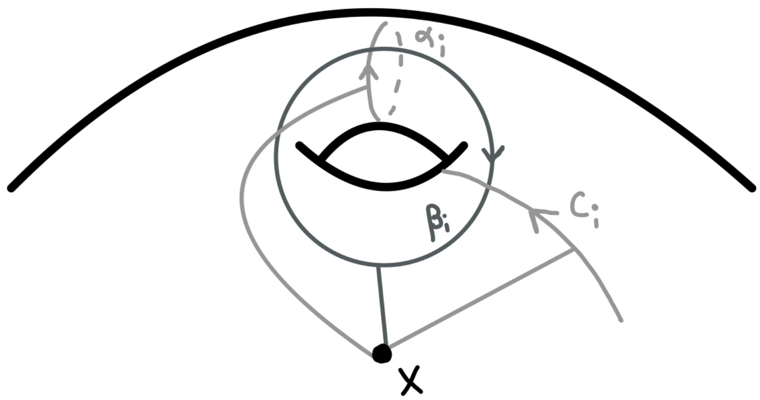
Let Σ_6 denotes the closed, connected, oriented surface of genus 6, and $\pi_1(\Sigma_6)$ its fundamental group. We denote the mapping class group of Σ_6 as $\text{Mod}(\Sigma_6)$. In [1], it is shown that three involutions are able to generate $\text{Mod}(\Sigma_6)$. These are denoted ρ_1, ρ_2, and $\rho_4 F_1$, where ρ_4 is a product of powers of ρ_1 and ρ_2, and F_1 is a product of 6 Dehn twists. We will refer to $\rho_4 F_1$ as ρ_3. We give a characterization of the fixed point sets of the pullbacks induced by these involutions on the representation variety $R(\Sigma_6, H_3)$ and the orbit space $\chi(\Sigma_6, H_3)$.

The figure on page 1 shows the construction and orientation of our surface. Define α_i, β_i, c_i as in that figure. The following:

$$\left\langle \alpha_1, \beta_1, \ldots, \alpha_6, \beta_6 \;\middle|\; \Pi_{i=1}^{6} [\alpha_i, \beta_i] \right\rangle$$

provides a finite presentation for $\pi_1(\Sigma_6)$. Note that each $c_i = \alpha_i^{-1} \alpha_{i+1}$. The c_i curves are defined on page 1.

Proceeding, all Dehn twists are *right* Dehn twists. This means that a curve α is affected by a twist about an unoriented loop γ as follows: α first approaches γ, then turns to the *right*, and follows γ before continuing along its original trajectory. A Dehn twist about an unoriented loop γ is denoted τ_γ, and compositions are understood as in function notation. Concatenation of two loops, such as $\alpha\beta$, is understood as "follow α first, then β". To view these curves as elements of the fundamental group, we must fix a base point. The base point x for our curves can be seen on page 1. We connect α_i to x by going around the *left* side of hole i. We connect β_i and c_i to x by a straight line. These connections may be seen in the figure below.



Below is a list of Dehn twist calculations that are used to compute concatenations of loops under each involution. Note that a_i denotes the unoriented parent curve of α_i, with the parents curves b_i for β_i, and c_i for c_i respectively. We have $\tau_{a_i}(\alpha_i) = \alpha_i$, $\tau_{b_i}(\beta_i) = \beta_i$ and $\tau_{c_i}(c_i) = c_i$. By convention we have that α_i^{-1}, β_i^{-1}, and c_i^{-1} also stay fixed under their respective parent Dehn twists. We also provide an algebraic computation table that shows how the involutions described in [1] act on α_i, β_i, and c_i.

Dehn Twist Calculations

$\tau_{a_i}(\beta_i) = \alpha_i^{-1}\beta_i$
$\tau_{b_i}(\alpha_i) = \alpha_i\beta_i$
$\tau_{b_i}(c_i) = \beta_i^{-1}c_i$
$\tau_{b_i}(c_{i-1}) = c_{i-1}\beta_i$
$\tau_{c_i}(\beta_i) = \beta_i c_i$
$\tau_{c_i}(\beta_{i+1}) = c_i^{-1}\beta_{i+1}$
$\tau_{a_i}^{-1}(\beta_i) = \alpha_i\beta_i$
$\tau_{b_i}^{-1}(\alpha_i) = \alpha_i\beta_i^{-1}$
$\tau_{b_i}^{-1}(c_i) = \beta_i c_i$
$\tau_{b_i}^{-1}(c_{i-1}) = c_{i-1}\beta_i^{-1}$
$\tau_{c_i}^{-1}(\beta_i) = \beta_i c_i^{-1}$
$\tau_{c_i}^{-1}(\beta_{i+1}) = c_i\beta_{i+1}$

Involution Calculations

	ρ_1	ρ_2	ρ_3
α_1	α_3^{-1}	α_2^{-1}	α_6^{-1}
β_1	β_3^{-1}	β_2^{-1}	$\alpha_6\beta_6^{-1}c_5$
c_1	c_2	c_1	c_5
α_2	α_2^{-1}	α_1^{-1}	α_5^{-1}
β_2	β_2^{-1}	β_1^{-1}	$c_5^{-1}\beta_5^{-1}$
c_2	c_1	c_6	$c_4\beta_4^{-1}$
α_3	α_1^{-1}	α_6^{-1}	$\alpha_4^{-1}\beta_4^{-1}$
β_3	β_1^{-1}	β_6^{-1}	β_4^{-1}
c_3	c_6	c_5	$\beta_4c_3\beta_3$
α_4	α_6^{-1}	α_5^{-1}	$\alpha_3^{-1}\beta_3$
β_4	β_6^{-1}	β_5^{-1}	β_3^{-1}
c_4	c_5	c_4	$\beta_3^{-1}c_2$
α_5	α_5^{-1}	α_4^{-1}	α_2^{-1}
β_5	β_5^{-1}	β_4^{-1}	$\beta_2^{-1}c_1^{-1}$
c_5	c_4	c_3	c_1
α_6	α_4^{-1}	α_3^{-1}	α_1^{-1}
β_6	β_4^{-1}	β_3^{-1}	$\alpha_1^{-1}c_1\beta_1^{-1}$
c_6	c_3	c_2	c_6

Fixed Points of the Induced Maps on $R(\Sigma_6, H_3)$

Since ρ_1, ρ_2, and ρ_3 are involutions in $\mathrm{Mod}(\Sigma_6)$, they are orientation-preserving homeomorphisms from Σ_6 to itself. From each involution, we get an induced automorphism on $\pi_1(\Sigma_6)$. This in turn induces a pullback from $R(\Sigma_6, H_3)$ to itself. We give a complete characterization of the fixed point sets of this pullback for each of ρ_1, ρ_2, and ρ_3. We will denote this fixed point set on the representation variety as $\mathrm{fix}_R(\rho_i)$. Similarly, denote the fixed point set on the orbit space as $\mathrm{fix}_\chi(\rho_i)$.

The Fixed Point Set $\mathbf{fix}_R(\rho_1)$

From our involution computation table, we see that $(A_1, ..., B_6) \in \text{fix}_R(\rho_1)$ if and only if:

$$
\begin{aligned}
A_1 &= A_3^{-1} \\
B_1 &= B_3^{-1} \\
A_2 &= A_2^{-1} \\
B_2 &= B_2^{-1} \\
&\;\;. \\
&\;\;. \\
&\;\;. \\
A_6 &= A_4^{-1} \\
B_6 &= B_4^{-1}
\end{aligned}
$$

$$\prod_{i=1}^{6}[A_i, B_i] = Id$$

Notice that $A_2 = A_2^{-1} \Leftrightarrow A_2 = Id$, and similarly $B_2 = Id$, $A_5 = Id$, and $B_5 = Id$. This simplifies the product of commutator condition. It then follows that A_3 and B_3 are determined by A_1 and B_1 respectively. Similarly, A_6 and B_6 are determined by A_4 and B_4. We can then fix A_1, B_1, A_4, and B_4 to determine the rest of the elements based on these. We can also rewrite the product of commutator condition entirely in terms of these elements. We then have:

$$\text{fix}_R(\rho_1) = \Big\{(A_1, B_1, Id, Id, A_1^{-1}, B_1^{-1}, A_4, B_4, Id, Id, A_4^{-1}, B_4^{-1}) \;\Big|\; [A_1, B_1][A_1^{-1}, B_1^{-1}][A_4, B_4][A_4^{-1}, B_4^{-1}] = Id\Big\}$$

However, this simplifies greatly:

$$\aleph(0, 0, 2a_{A_1}b_{B_1} - 2a_{B_1}b_{A_1}) \cdot \aleph(0, 0, 2a_{A_4}b_{B_4} - 2a_{B_4}b_{A_4}) = \aleph(0, 0, 0)$$

This is equivalent to: $2a_{A_1}b_{B_1} - 2a_{B_1}b_{A_1} + 2a_{A_4}b_{B_4} - 2a_{B_4}b_{A_4} = 0$ which gives $a_{A_1}b_{B_1} - a_{B_1}b_{A_1} + a_{A_4}b_{B_4} - a_{B_4}b_{A_4} = 0$ and thus:

$$\aleph(0, 0, a_{A_1}b_{B_1} - a_{B_1}b_{A_1}) \cdot \aleph(0, 0, a_{A_4}b_{B_4} - a_{B_4}b_{A_4}) = \aleph(0, 0, 0)$$

We then have the simplified product of commutator condition: $[A_1, B_1][A_4, B_4] = Id$. Thus the fixed point set is:

$$\mathrm{fix}_R(\rho_1) = \Big\{(A_1, B_1, Id, Id, A_1^{-1}, B_1^{-1}, A_4, B_4, Id, Id, A_4^{-1}, B_4^{-1}) \ \Big| \ [A_1, B_1][A_4, B_4] = Id\Big\}$$

The Fixed Point Set $\mathbf{fix}_R(\rho_2)$

The same strategy is used to compute $\mathrm{fix}_R(\rho_2)$. Fix half of the A_i and half of the B_i, and determine the rest based on these. This gives:

$$\mathrm{fix}_R(\rho_2) = \{(A_1, B_1, A_1^{-1}, B_1^{-1}, A_3, B_3, A_4, B_4, A_4^{-1}, B_4^{-1}, A_3^{-1}, B_3^{-1}) \ \Big| \ [A_1, B_1][A_1^{-1}, B_1^{-1}][A_3, B_3][A_4, B_4][A_4^{-1}, B_4^{-1}][A_3^{-1}, B_3^{-1}] = Id\Big\}$$

This again simplifies, giving $[A_1, B_1][A_3, B_3][A_4, B_4] = Id$. Thus our fixed point set is:

$$\mathrm{fix}_R(\rho_2) = \Big\{(A_1, B_1, A_1^{-1}, B_1^{-1}, A_3, B_3, A_4, B_4, A_4^{-1}, B_4^{-1}, A_3^{-1}, B_3^{-1}) \ \Big| \ [A_1, B_1][A_3, B_3][A_4, B_4] = Id\Big\}$$

The Fixed Point Set $\mathbf{fix}_R(\rho_3)$

Notice from the involution calculation table that some generators are sent to products involving the c_i loops. We rewrite these to be in terms of the α_i's. We then rewrite every product to be in terms of α_i and β_i for $i \in \{1, 2, 3\}$. This gives the following table for the induced map ρ_3^*:

	ρ_3^*
A_1	A_1
B_1	$A_1^{-1}B_1A_2^{-1}A_1^2A_2A_1^{-1}$
A_2	A_2
B_2	$A_1A_2^{-1}A_1^{-1}A_2B_2$
A_3	$B_3^{-1}A_3B_3$
B_3	B_3
A_4	$A_3^{-1}B_3$
B_4	B_3^{-1}
A_5	A_2^{-1}
B_5	$B_2^{-1}A_2^{-1}A_1$
A_6	A_1^{-1}
B_6	$A_1^{-2}A_2B_1^{-1}$

Since each matrix must equal its image, the equation for B_2 gives that A_1 and A_2 commute. We then have that A_1 and B_1 commute, forcing A_6 and B_6 to commute. It also follows that A_3 and B_3 commute, forcing A_4 and B_4 to commute. The product of commutator condition reduces to $[A_2, B_2][A_5, B_5] = Id$. This further reduces to $[A_2, B_2] = [B_2^{-1}, A_2^{-1}]$. This says that $\aleph(0, 0, a_{A_2}b_{B_2} - a_{B_2}b_{A_2}) = \aleph(0, 0, a_{B_2}b_{A_2} - a_{A_2}b_{B_2})$. So then $2a_{A_2}b_{B_2} = 2a_{B_2}b_{A_2}$ which gives $a_{A_2}b_{B_2} = a_{B_2}b_{A_2}$, so $[A_2, B_2] = Id$. This implies $[A_5, B_5] = Id$, eliminating the product of commutators condition. Since A_{1-3} and B_{1-3} determine A_{4-6} and B_{4-6}, we have:

$$\det \begin{bmatrix} a_{A_i} & b_{A_i} \\ a_{B_i} & b_{B_i} \end{bmatrix} = 0$$

for $i \in \{1, 2, 3\}$ being a necessary and sufficient condition. We then have the fixed point set:

$$\text{fix}_R(\rho_3) = \Big\{(A_1, B_1, A_2, B_2, A_3, B_3, A_3^{-1}B_3, B_3^{-1}, A_2^{-1},$$

$$B_2^{-1}A_2^{-1}A_1, A_1^{-1}, A_1^{-2}A_2B_1^{-1}) \;\Big|\; \det \begin{bmatrix} a_{A_i} & b_{A_i} \\ a_{B_i} & b_{B_i} \end{bmatrix} = 0 \text{ for all } i \in \{1, 2, 3\}\Big\}$$

Fixed Points of the Induced Maps on $\chi(\Sigma_6, H_3)$

We now undergo the same procedure on the orbit space $\chi(\Sigma_6, H_3)$. Since the orbit space introduces simultaneous conjugation, the computations are more involved.

The Fixed Point Set $\mathbf{fix}_\chi(\rho_1)$

The simultaneous conjugation action gives the following system of 12 equations:

$$
\begin{aligned}
gA_1g^{-1} &= A_3^{-1} \\
gB_1g^{-1} &= B_3^{-1} \\
gA_2g^{-1} &= A_2^{-1} \\
gB_2g^{-1} &= B_2^{-1} \\
gA_3g^{-1} &= A_1^{-1} \\
gB_3g^{-1} &= B_1^{-1} \\
gA_4g^{-1} &= A_6^{-1} \\
gB_4g^{-1} &= B_6^{-1} \\
gA_5g^{-1} &= A_5^{-1} \\
gB_5g^{-1} &= B_5^{-1} \\
gA_6g^{-1} &= A_4^{-1} \\
gB_6g^{-1} &= B_4^{-1}
\end{aligned}
$$

Equations 1 and 5 give $g^2 \in C_{A_1}$ and $g^2 \in C_{A_3}$. From lemma 2, it then follows that $g \in C_{A_1}$ and $g \in C_{A_3}$. By similar methodology we have $g \in C_{A_i}$ and $g \in C_{B_i}$ for all $i \in \{1, 2, 3, 4, 5, 6\}$. Recall that centralizers are symmetric: If $a \in C_b$, then $b \in C_a$. So all 12 of our matrices are in C_g.

This gives $[f] = [A_1, ..., B_6]$ satisfying the above equations that imply $A_i, B_i \in C_g$. This means that f satisfies these equations with $g = Id$, giving $f \in \text{fix}_R(\rho_1)$.

The Fixed Point Set $\mathbf{fix}_\chi(\rho_2)$

We have the following system of 12 equations:

$$\begin{aligned}
gA_1g^{-1} &= A_2^{-1}\\
gB_1g^{-1} &= B_2^{-1}\\
gA_2g^{-1} &= A_1^{-1}\\
gB_2g^{-1} &= B_1^{-1}\\
gA_3g^{-1} &= A_6^{-1}\\
gB_3g^{-1} &= B_6^{-1}\\
gA_4g^{-1} &= A_5^{-1}\\
gB_4g^{-1} &= B_5^{-1}\\
gA_5g^{-1} &= A_4^{-1}\\
gB_5g^{-1} &= B_4^{-1}\\
gA_6g^{-1} &= A_3^{-1}\\
gB_6g^{-1} &= B_3^{-1}
\end{aligned}$$

We can again derive $g^2 \in C_{A_i}$ and $g^2 \in C_{B_i}$, and thus $g \in C_{A_i}$ and $g \in C_{B_i}$ for all $i \in \{1, 2, 3, 4, 5, 6\}$ by lemma 2. So each A_i and B_i are in C_g, and by the same argument used for $\text{fix}_\chi(\rho_1)$, we have $f \in \text{fix}_R(\rho_2)$.

The Fixed Point Set $\mathbf{fix}_\chi(\rho_3)$

After producing a set of 12 equations, lemma 2 gives $g \in C_{A_1}$, $g \in C_{A_2}$, $g \in C_{B_3}$, $g \in C_{B_4}$, $g \in C_{A_5}$, and $g \in C_{A_6}$. We then proceed to rewrite our 12 equations entirely in terms of A_i and B_i for $i \in \{1, 2, 3\}$:

$$
\begin{aligned}
gA_1g^{-1} &= g^{-1}A_1g \\
gB_1g^{-1} &= g^{-1}A_1^{-1}B_1A_2^{-1}A_1^2A_2A_1^{-1}g \\
gA_2g^{-1} &= g^{-1}A_2g \\
gB_2g^{-1} &= g^{-1}A_1A_2^{-1}A_1^{-1}A_2B_2g \\
gA_3g^{-1} &= g^{-1}B_3^{-1}A_3B_3g \\
gB_3g^{-1} &= g^{-1}B_3g \\
gA_4g^{-1} &= A_3^{-1}B_3 \\
gB_4g^{-1} &= B_3^{-1} \\
gA_5g^{-1} &= A_2^{-1} \\
gB_5g^{-1} &= B_2^{-1}A_2^{-1}A_1 \\
gA_6g^{-1} &= A_1^{-1} \\
gB_6g^{-1} &= A_1^{-2}A_2B_1^{-1}
\end{aligned}
$$

As a consequence of $g \in C_{A_1}$ and $g \in C_{A_2}$, proposition 3 gives that A_1 and A_2 commute. Equation 4 immediately gives $g \in C_{B_2}$ by lemma 2. From the commutativity of A_1 and A_2 we can then simplify the second equation to get $gB_1g^{-1} = g^{-1}A_1^{-1}B_1A_1g$. We then reach $A_1g^2 \in C_{B_1}$. Notice that $A_1g^2 = \aleph(a_A + 2a_g, b_A + 2b_g, c_A + a_A 2b_g + a_g b_g + 2c_g)$. Computing $A_1g^2B_1$ and $B_1A_1g^2$, we see that they are equal if and only if $a_{B_1}b_{A_1} + a_{B_1}2b_g = a_{A_1}b_{B_1} + 2a_g b_{B_1}$. Since $g \in C_{A_1}$ and $A_1 \in C_g$, we have $a_{A_1} = ka_g$ and $b_{A_1} = kb_g$. This gives $a_{B_1}kb_g + a_{B_1}2b_g = ka_g b_{B_1} + 2a_g b_{B_1}$, so $a_{B_1}b_g(k+2) = a_g b_{B_1}(k+2)$, and thus $a_{B_1}b_g = a_g b_{B_1}$. This implies $a_{B_1}b_g - a_g b_{B_1} = 0$, and thus $\aleph(0, 0, a_{B_1}b_g - a_g b_{B_1}) = \aleph(0, 0, 0)$. This gives that $[B_1, g] = Id$, so $g \in C_{B_1}$. The same argument on equation 5 gives $g \in C_{A_3}$. We can then use the commutativity of A_3 and B_3 to get $g \in C_{A_4}$. We can then derive $g \in C_{B_5}$ and $g \in C_{B_6}$. Remarkably, all of our elements commute, and each A_i and B_i are again in C_g. We use the same argument from the last two sections, and have $f \in \text{fix}_R(\rho_3)$.

Topology of the Various Fixed Point Sets

We begin with a useful corollary that follows from the fixed point sets attained above.

Corollary 1. *The quotient map* $\Psi : \text{fix}_R(\rho_i) \to \text{fix}_\chi(\rho_i)$ *for each of* ρ_1*,* ρ_2*, and* ρ_3 *is surjective.*

Proof. We construct a surjection of the form $\Psi : \text{fix}_R(\rho_i) \to \text{fix}_\chi(\rho_i)$ by $\Psi : f \mapsto [f]$ for any $f \in \text{fix}_R(\rho_i)$. In each of our orbit space calculations, g commuted with each element by necessity. This causes a reduction to the representation variety $R(\Sigma_g, H_3)$. The map Ψ is exactly the quotient map that defines $\chi(\Sigma_g, H_3)$, and is evident from the above calculations. □

For ρ_1, ρ_2, and ρ_3 we show that $\text{fix}_R(\rho_i)$ is contractible, and $\text{fix}_\chi(\rho_i)$ is connected. This is presented as a series of three theorems.

Theorem 3. $\text{fix}_R(\rho_1)$ *is contractible, and* $\text{fix}_\chi(\rho_1)$ *is connected.*

Proof. From our derivation of $\text{fix}_R(\rho_1)$, it follows that:

$$\text{fix}_R(\rho_1) \cong \left\{(A_1, B_1, A_4, B_4) \;\middle|\; [A_1, B_1][A_4, B_4] = Id\right\} \cong R(\Sigma_2, H_3)$$

From theorem 2, this space is contractible. From the existence of Ψ by corollary 1, it follows that $\text{fix}_\chi(\rho_1)$ is connected. □

Theorem 4. $\text{fix}_R(\rho_2)$ *is contractible, and* $\text{fix}_\chi(\rho_2)$ *is connected.*

Proof. From our derivation of $\text{fix}_R(\rho_2)$, it follows that $\text{fix}_R(\rho_2) \cong R(\Sigma_3, H_3)$ since both are of the form:

$$\left\{(A_1, B_1, A_3, B_3, A_4, B_4) \;\middle|\; [A_1, B_1][A_3, B_3][A_4, B_4] = Id\right\}$$

From theorem 2, this space is contractible. From the existence of Ψ by corollary 1, it follows that $\text{fix}_\chi(\rho_2)$ is connected. □

Theorem 5. $\text{fix}_R(\rho_3)$ *is contractible, and* $\text{fix}_\chi(\rho_3)$ *is connected.*

Proof. From our derivation of $\text{fix}_R(\rho_3)$, it follows that $\text{fix}_R(\rho_3) \cong R(\Sigma_1, H_3)^3$ since $\text{fix}_R(\rho_3)$ is determined by three pairs A_i, B_i with $[A_i, B_i] = Id$ for each pair individually. Since $R(\Sigma_1, H_3)$ is contractible, so is $R(\Sigma_1, H_3)^3$, and thus $\text{fix}_R(\rho_3)$ is contractible. It then follows that $\text{fix}_\chi(\rho_3)$ is connected. □

Concluding Thoughts

While this document only provides an introduction to the theory of Representation Varieties, Character Varieties, and Orbit Spaces, these ideas extend deeply in Geometric Topology. Representation and character varieties arise in the growing fields of Gauge Theory and Chern–Simons Theory. For the interested reader, see [2].

References

[1] O. Yildiz, *Generating the Mapping Class Group by Three Involutions.* Arxiv Preprint (2020).

[2] J. Marche, *Geometry of Representation Spaces in SU(2).* Arxiv Preprint (2010).