A nontrivial sphere bundle in U(2) coming from the trace

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1 Introduction

Let X be the set of $A \in \mathrm{U}(2)$ with trace zero.

Theorem 1.1. *The determinant map $X \to S^1$ is a non-trivial S^2-bundle. More specifically, this bundle is isomorphic to the mapping torus of the map $\phi : S^2 \to S^2$ given by $\phi(x, y, z) = (x, y, -z)$.*

Proof. View $S^3 \subseteq \mathbb{C}^2$ as a subset of $\mathbb{C}^2$ and consider the map

$$\begin{array}{rcl} \Phi : S^1 \times S^3 & \longrightarrow & \mathrm{U}(2) \\ (\delta, (\alpha, \beta)) & \longmapsto & \begin{pmatrix} \delta\alpha & \delta\beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \end{array}$$

This map is a diffeomorphism with inverse given by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \Big(\det(A), \Big(\det(A)^{-1}a, \det(A)^{-1}b\Big)\Big).$$

It is clear also that Φ is a bundle isomorphism relative to the projection on $S^1 \times S^3$ to S^1 and the determinant map on $\mathrm{U}(2)$.

The trace map on $\mathrm{U}(2)$ pulls back under Φ to have the form

$$(\delta, (\alpha, \beta)) \longmapsto \delta\alpha + \overline{\alpha}.$$

It follows that Φ identifies X with the subset Y of $(\delta, (\alpha, \beta)) \in S^1 \times S^3$ with $\delta\alpha + \overline{\alpha} = 0$. That is,

$$Y = \sqcup_{\delta \in S^1} S^2_\delta,$$

where $S^2_\delta := S^3 \cap \ell_\delta \times \mathbb{C}$, where

$$\ell_\delta := \{\alpha \in \mathbb{C} \mid \delta\alpha + \overline{\alpha} = 0\}.$$

Clearly S^2_δ is an equatorial 2-sphere in S^3, since ℓ_δ is a line through the origin in $\mathbb{C}$. Consider the bundle $\mathcal{L} \to S^1$ with fiber ℓ_δ over δ. This is the Möbius bundle (e.g., write $\delta = \eta^2$, then $\alpha \in \ell_{\eta^2}$ if and only if $\eta\alpha$ is purely imaginary; that is, ℓ_δ is obtained by from the imaginary axis by rotating by η^{-1}). Tracing through the identifications, the result follows. □