The osculating helix

David L. Duncan

1 Helixes, or helices, or whatever

View $\mathbb{R}^3$ as consisting of column vectors, so matrix multiplication works out as we like it to. Let's say that a *helix in standard form* is a curve $\mathbf{h}_{a,b} : \mathbb{R} \to \mathbb{R}^3$ of the form

$$\mathbf{h}_{a,b}(s) = \begin{pmatrix} a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ b\frac{s}{\sqrt{a^2+b^2}} \end{pmatrix},$$

where $(a, b) \in [0, \infty) \times \mathbb{R}$ is fixed and non-zero. When a and b are clear from context, they will be dropped from the notation: $\mathbf{h} := \mathbf{h}_{a,b}$.

Note that

$$\mathbf{h}'(s) = \begin{pmatrix} -\frac{a}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ \frac{a}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ \frac{b}{\sqrt{a^2+b^2}} \end{pmatrix},$$

has unit norm, so this helix is parametrized by arclength, and thus this is the unit tangent vector

$$\mathbf{T}_{\mathbf{h}}(s) := \mathbf{h}'(s).$$

The other terms of the Frenet frame are given by

$$\begin{aligned} \mathbf{N}_{\mathbf{h}}(s) := \frac{\mathbf{T}'_{\mathbf{h}}(s)}{\|\mathbf{T}'_{\mathbf{h}}(s)\|} &= \begin{pmatrix} -\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ 0 \end{pmatrix}, \\ \mathbf{B}_{\mathbf{h}}(s) := \mathbf{T}_{\mathbf{h}}(s) \times \mathbf{N}_{\mathbf{h}}(s) &= \begin{pmatrix} \frac{b}{\sqrt{a^2+b^2}}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ -\frac{b}{\sqrt{a^2+b^2}}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) \\ \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}. \end{aligned}$$

At $s = 0$ this frame takes the from

$$\begin{aligned}
\mathbf{T_h}(0) &= \begin{pmatrix} 0 \\ \frac{a}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} \end{pmatrix}, \\
\mathbf{N_h}(0) &= \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \\
\mathbf{B_h}(0) &= \begin{pmatrix} 0 \\ -\frac{b}{\sqrt{a^2+b^2}} \\ \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}.
\end{aligned} \tag{1}$$

The curvature κ and torsion τ of $\mathbf{h}$ are both constant in s and given by

$$\kappa = \frac{a}{a^2+b^2}, \qquad \text{and} \qquad \tau = \frac{b}{a^2+b^2}. \tag{2}$$

Note that for any non-zero tuple $(\kappa, \tau) \in [0, \infty) \times \mathbb{R}$, there is a unique non-zero tuple $(a, b) \in [0, \infty) \times \mathbb{R}$ so that (2) holds; indeed,

$$a = \frac{\kappa}{\kappa^2+\tau^2}, \qquad b = \frac{\tau}{\kappa^2+\tau^2}. \tag{3}$$

More generally, a *helix* is a function of the form

$$\mathbf{H} := \mathbf{H}_{a,b,L} := L \circ \mathbf{h}_{a,b},$$

where $\mathbf{h}_{a,b}$ is a helix in standard form, and $L : \mathbb{R}^3 \to \mathbb{R}^3$ is an orientation-preserving affine-linear isometry. This latter condition means that L is of the from

$$L(\mathbf{v}) = \mathbf{v}_0 + R\mathbf{v}$$

for some fixed vector $\mathbf{v}_0 \in \mathbb{R}^3$ and some linear transformation $R : \mathbb{R}^3 \to \mathbb{R}^3$ satisfying $R^T = R$ and $\det(R) > 0$ (from which it follows that $\det(R) = 1$). The helix $\mathbf{H}_{a,b,L}$ is also parametrized by arclength, and the Frenet frame is given by

$$R\mathbf{T_h}(s), \qquad R\mathbf{N_h}(s), \qquad R\mathbf{B_h}(s).$$

The curvature and torsion of $\mathbf{H}_{a,b,L}$ are also constant in s and given by (2).

2 The osculating helix

Let $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ be a curve parametrized by arclength. Assume that the curvature κ and torsion τ of $\mathbf{r}$ are not both zero at time $s = 0$. We will show that there is a helix $\mathbf{H} = \mathbf{H}_{a,b,L}$ so that, at $s = 0$, the helix $\mathbf{H}$ (nearly) agrees with $\mathbf{r}$ to third order in the following sense:

$$\mathbf{r}(0) = \mathbf{H}(0) \tag{4}$$
$$\mathbf{r}'(0) = \mathbf{H}'(0) \tag{5}$$
$$\mathbf{r}''(0) = \mathbf{H}''(0) \tag{6}$$
$$\operatorname{proj}_{\mathbf{N}(0)^\perp}\left(\mathbf{r}'''(0)\right) = \mathbf{H}'''(0). \tag{7}$$

Here $\operatorname{proj}_{\mathbf{N}(0)^\perp}$ is the orthogonal projection to the plane normal to $\mathbf{N}(0)$; thus, part of the claim of (7) is that $\mathbf{H}'''(0)$ lies in this plane.

To carry this out, we need to define a, b, and L, and then check that the above properties hold. Define a, b in terms of $\kappa(0), \tau(0)$ by

$$a = \frac{\kappa(0)}{\kappa(0)^2 + \tau(0)^2}, \qquad b = \frac{\tau(0)}{\kappa(0)^2 + \tau(0)^2}.$$

Of course, this is just (3). From these, construct the standard form helix $\mathbf{h} = \mathbf{h}_{a,b}$. To define $L(\mathbf{v}) = \mathbf{v}_0 + R\mathbf{v}$, write $\mathbf{T}, \mathbf{N}, \mathbf{B}$ for the Frenet frame of $\mathbf{r}$. Define R to be the linear transformation associated to the change of basis from $\mathbf{T}_\mathbf{h}(0), \mathbf{N}_\mathbf{h}(0), \mathbf{B}_\mathbf{h}(0)$ to $\mathbf{T}(0), \mathbf{N}(0), \mathbf{B}(0)$; thus

$$R(\mathbf{T}_\mathbf{h}(0)) = \mathbf{T}(0), \qquad R(\mathbf{N}_\mathbf{h}(0)) = \mathbf{N}(0), \qquad R(\mathbf{B}_\mathbf{h}(0)) = \mathbf{B}(0). \tag{8}$$

Since both Frenet frames (the one for the helix in standard form, and the one for $\mathbf{r}$) are orthonormal and oriented, it follows that R is symmetric with positive determinant, as required. Next, take

$$\mathbf{v}_0 := \mathbf{r}(0) - R(\mathbf{h}(0)).$$

This defines our helix $\mathbf{H} = \mathbf{H}_{a,b,L}$, so it suffices to verify that it satisfies (4—7).

Verification of (4): The definition of our helix is that $\mathbf{H} = \mathbf{v}_0 + R \circ \mathbf{h}$. Then the definition of $\mathbf{v}_0$ gives

$$\mathbf{r}(0) = R(\mathbf{h}(0)) + \mathbf{v}_0 = \mathbf{H}(0).$$

Verification of (5): Since $\mathbf{r}$ and $\mathbf{H}$ are both parametrized by arclength, we have

$$\mathbf{r}'(0) = \mathbf{T}(0) = R(\mathbf{T}_\mathbf{h}(0)) = R(\mathbf{h}'(0)) = \frac{d}{ds}\Big|_{s=0} \mathbf{v}_0 + R(\mathbf{h}(s)) = \mathbf{H}'(0).$$

Verification of (6): The quick version is that $\mathbf{r}$ and $\mathbf{H}$ have the same normal vector at $s = 0$, by construction. They also have the same curvature by construction. Then (6) follows because the second derivative of a unit-speed curve is the curvature times the normal vector.

For those who prefer equations over words, here are some of the former:

$$
\begin{aligned}
\mathbf{r}''(0) &= \mathbf{T}'(0) \\
&= \kappa(0)\mathbf{N}(0) \\
&= \frac{a}{a^2+b^2}\mathbf{N}(0) \\
&= \frac{a}{a^2+b^2}R(\mathbf{N}_{\mathbf{h}}(0)) \\
&= R\left(\frac{a}{a^2+b^2}\mathbf{N}_{\mathbf{h}}(0)\right) \\
&= R(\mathbf{T}_{\mathbf{h}}'(0)) \\
&= R(\mathbf{h}''(0)) \\
&= \mathbf{H}''(0).
\end{aligned}
$$

Verification of (7): Differentiate

$$\mathbf{r}'' = \kappa\mathbf{N}$$

in s to get

$$\mathbf{r}''' = \kappa'\mathbf{N} + \kappa\mathbf{N}' = \kappa'\mathbf{N} - \kappa^2\mathbf{T} + \kappa\tau\mathbf{B}.$$

where we used the Frenet–Serret formula $\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}$. Similarly,

$$\mathbf{H}''' = -\frac{a^2}{(a^2+b^2)^2}R(\mathbf{T}_{\mathbf{h}}) + \frac{ab}{(a^2+b^2)^2}R(\mathbf{B}_{\mathbf{h}})$$

since the curvature of the helix is constant. At time $s = 0$, we have therefore obtain

$$
\begin{aligned}
\mathbf{r}'''(0) &= \kappa'\mathbf{N}(0) - \kappa(0)^2\mathbf{T}(0) + \kappa(0)\tau(0)\mathbf{B}(0) \\
\mathbf{H}'''(0) &= -\kappa(0)^2\mathbf{T}(0) + \kappa(0)\tau(0)\mathbf{B}(0)
\end{aligned}
$$

The claim of (7) now follows.

3 A more explicit formula for H

In the above, we gave a general formula for the osculating helix of $\mathbf{r}$ at time 0. More generally, the osculating helix for $\mathbf{r}$ at time t is given by

$$\mathbf{H}(s,t) := \mathbf{r}(t) - R_t(\mathbf{h}_{a,b}(0)) + R_t(\mathbf{h}_{a,b}(s)),$$

where a, b are computed via (3) from the curvature $\kappa(t)$ and $\tau(t)$ of $\mathbf{r}$ at t, and R_t is defined by

$$R_t(\mathbf{T}_{\mathbf{h}}(0)) = \mathbf{T}(t), \qquad R_t(\mathbf{N}_{\mathbf{h}}(0)) = \mathbf{N}(t), \qquad R_t(\mathbf{B}_{\mathbf{h}}(0)) = \mathbf{B}(t),$$

which is just (8), but shifted to $s = t$ for the curve $\mathbf{r}$. The goal here is to give a more explicit formula for $\mathbf{H}$, and this comes down to giving a more explicit formula for R_t.

Let $R_{\mathbf{r}(t)}$ be the linear transformation taking the standard basis e_1, e_2, e_3 to the Frenet frame $\mathbf{T}(t), \mathbf{N}(t), \mathbf{B}(t)$ for $\mathbf{r}$ at t. Thus, the matrix for $R_{\mathbf{r}(t)}$ is

$$[R_{\mathbf{r}(t)}] = \begin{pmatrix} | & | & | \\ \mathbf{T}(t) & \mathbf{N}(t) & \mathbf{B}(t) \\ | & | & | \end{pmatrix}$$

Likewise, let $R_{\mathbf{h}(0)}$ be the linear transformation taking the standard basis e_1, e_2, e_3 to the Frenet frame $\mathbf{T_h}(0), \mathbf{N_h}(0), \mathbf{B_h}(0)$ for $\mathbf{h}$ at time 0. By (3), the matrix for $R_{\mathbf{h}(0)}$ is

$$[R_{\mathbf{h}(0)}] = \begin{pmatrix} | & | & | \\ \mathbf{T_h}(0) & \mathbf{N_h}(0) & \mathbf{B_h}(0) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ \frac{a}{\sqrt{a^2+b^2}} & 0 & -\frac{b}{\sqrt{a^2+b^2}} \\ \frac{b}{\sqrt{a^2+b^2}} & 0 & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}$$

Note that this is orthogonal, so its inverse is given by its transpose.

Then $R_t = R_{\mathbf{r}(t)} \circ R_{\mathbf{h}}^{-1}$ and the matrix for R_t is given by

$$[R_t] = \begin{pmatrix} | & | & | \\ \mathbf{T}(t) & \mathbf{N}(t) & \mathbf{B}(t) \\ | & | & | \end{pmatrix} \begin{pmatrix} 0 & \frac{a}{\sqrt{a^2+b^2}} & \frac{b}{\sqrt{a^2+b^2}} \\ -1 & 0 & 0 \\ 0 & -\frac{b}{\sqrt{a^2+b^2}} & \frac{a}{\sqrt{a^2+b^2}} \end{pmatrix}$$

Carrying out the multiplication, we find that the columns of this matrix $[R_t]$ are

$$-\mathbf{N}(t), \qquad \frac{a}{\sqrt{a^2+b^2}}\mathbf{T}(t) - \frac{b}{\sqrt{a^2+b^2}}\mathbf{B}(t), \qquad \frac{b}{\sqrt{a^2+b^2}}\mathbf{T}(t) + \frac{a}{\sqrt{a^2+b^2}}\mathbf{B}(t).$$

We thus arrive at a formula for $\mathbf{H}$:

$$\begin{aligned} \mathbf{H}(s,t) &= \mathbf{r}(t) + \frac{a^2\sqrt{a^2+b^2}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right) + b^2 s}{a^2+b^2}\mathbf{T}(t) \\ &\quad + a\left(1 - \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\right)\mathbf{N}(t) + ab\frac{s - \sqrt{a^2+b^2}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)}{a^2+b^2}\mathbf{B}(t) \end{aligned}$$

The quantities a, b depend on κ and τ, and are thus t-dependent. Expressing $\mathbf{H}$

in terms of κ and τ, we get

$$\begin{aligned}\mathbf{H}(s,t) &= \mathbf{r} + \frac{1}{(\kappa^2+\tau^2)^{3/2}}\left(s\tau^2\sqrt{\kappa^2+\tau^2} + \kappa^2 \sin\left(s\sqrt{\kappa^2+\tau^2}\right)\right)\mathbf{T} \\ &\quad + \frac{\kappa}{\kappa^2+\tau^2}\left(1-\cos\left(s\sqrt{\kappa^2+\tau^2}\right)\right)\mathbf{N} \\ &\quad + \kappa\tau\frac{s\sqrt{\kappa^2+\tau^2} - \sin\left(s\sqrt{\kappa^2+\tau^2}\right)}{(\kappa^2+\tau^2)^{3/2}}\mathbf{B}\end{aligned}$$

where the t-dependence has been suppressed in the right to save space.

4 Polynomial approximations

It is entertaining to consider successive Maclaurin polynomial approximations $\mathbf{p}_k$ of $\mathbf{H}$ in s, which we can readily read off from the Maclaurin approximations of sine and cosine. The first four approximations are as follows

$$\begin{aligned}\mathbf{p}_0(s,t) &= \mathbf{r}(t) \\ \mathbf{p}_1(s,t) &= \mathbf{r}(t) + s\mathbf{T}(t) \\ \mathbf{p}_2(s,t) &= \mathbf{r}(t) + s\mathbf{T}(t) + \tfrac{1}{2}\kappa(t)s^2\mathbf{N}(t) \\ \mathbf{p}_3(s,t) &= \mathbf{r}(t) + \left(s - \tfrac{1}{6}\kappa(t)^2 s^3\right)\mathbf{T}(t) + \tfrac{1}{2}\kappa(t)s^2\mathbf{N}(t) + \tfrac{1}{6}\kappa(t)\tau(t)s^3\mathbf{B}(t)\end{aligned}$$

The quadratic $\mathbf{p}_2$ is the osculating parabola[1] The cubic $\mathbf{p}_3$ is what we would expect an "osculating cubic" to be.

[1]To match with the paper, take $u = \kappa(t)s$. This difference arises because I have parametrized my helix to be unit speed while, in the paper, the osculating circle has speed $1/\kappa$.