The Hodge star on the 3-sphere

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Consider the 3-sphere $S^3 \subseteq \mathbb{R}^4$ with the standard metric and coordinates $p = (x, y, z, w)$. Write $*_{S^3} : \Lambda^k T^* S^3 \to \Lambda^{3-k} T^* S^3$ for the Hodge star. The goal of this document is to show that

$$*_{S^3}(dx \wedge dy) = \sqrt{(1-x^2)(1-y^2)}\, \frac{zdw - wdz}{\sqrt{z^2+w^2}} \in T_p^* S^3. \tag{1}$$

There are plenty of examples floating around for computing the Hodge star of forms on vector spaces, and also of 0-, 1-, and top-degree-forms on manifolds, but far fewer of a middle-dimensional form on a (non-vector space) manifold. The broader point of this note is to illustrate some techniques for computing the Hodge star of a higher degree form on a manifold that isn't a vector space.

Remark 0.1. *(a) The formula in (1) has extensions to other spheres. For example, using standard coordinates (x, y, z) on S^2 we have*

$$*_{S^2} dx = zdy - ydz$$

while on S^1 we have

$$*_{S^1} 1 = xdy - ydx.$$

More generally, using coordinates $x_1, \ldots, x_n, x_{n+1}$ on S^n, we have

$$\begin{aligned} &*_{S^n}(dx_1 \wedge \ldots \wedge dx_{n-1}) \\ &\qquad = (-1)^{n+1} \sqrt{(1-x_1^2)(1-x_2^2)\ldots(1-x_{n-1}^2)}\, \frac{x_n dx_{n+1} - x_{n+1} dx_n}{\sqrt{x_n^2 + x_{n+1}^2}}. \end{aligned}$$

I will leave it up to you to verify this formula for $n \neq 3$.

(b) Note that the 2-form $dx \wedge dy$ is smooth, so $*_{S^3}(dx \wedge dy)$ should be smooth as well. However, the right side of (1) sure seems like it has a serious issue when $z = w = 0$. But fret-not dear reader: this is just an artifact of the coordinates, as I will now try to convince you. Write r, θ for the typical polar coordinates in the zw-plane (so $r^2 = z^2 + w^2$). In terms of these coordinates, we have $(z^2 + w^2)^{-1/2}(zdw - wdz) = rd\theta$ extends continuously across $r = 0$. Thus, the right side of (1) is continuous when $z = w = 0$. It is in fact smooth too, and passing to local coordinates would show this more clearly.

(c) On T^*S^3 we have

$$xdx + ydy + zdz + wdw = 0$$

which is a consequence of differentiating the defining equation $x^2 + y^2 + z^2 + w^2 = 1$. Using these equations, the right side of (1) can be written in many different ways. One benefit of the way it is currently written is that it treats the variables z and w (anti-)symmetrically.

Before getting to the proof, I will briefly recall the a few facts about the Hodge star. Suppose E is an n-dimensional real vector space equipped with an orientation and an inner product $\langle \cdot, \cdot \rangle$. This data determines a volume form $d\text{vol}_E$ as well as an inner product (also denoted $\langle \cdot, \cdot \rangle$) on the alternating product $\Lambda^k E$ for each k. Then the *Hodge star* on E is the linear map $* : \bigoplus_k \Lambda^k E \to \bigoplus_k \Lambda^k E$ with the property that if $v \in \Lambda^k E$, then $*v \in \Lambda^{n-k} E$ is the unique multivector satisfying

$$w \wedge *v = \langle w, v \rangle d\text{vol} \tag{2}$$

for all $w \in \Lambda^k E$.

Now suppose M is an oriented n-manifold with a Riemannian metric. Then each cotangent space $T_p^* M$ is an oriented vector space with an inner product and so admits a Hodge star. This varies smoothly with the basepoint p and so the pointwise Hodge star determines a bundle map

$$*_M : \Lambda^k T^* M \longrightarrow \Lambda^{n-k} T^* M$$

covering the identity. This bundle map is called the *Hodge star* on M.

Finally, before getting to the proof of (1), I want to highlight a subtlety that is hidden in the way I have written (1). The 1-forms dx, dy, dz, dw appearing in are technically 1-forms on $\mathbb{R}^4$, so the forms dx, dy etc., that appear in (1) really mean their pull back (restriction) to S^3. That is, letting $\iota : S^3 \to \mathbb{R}^4$ be the inclusion, we are really looking to compute a formula for the 1-form $*_{S^3}(\iota^*(dx \wedge dy))$ in terms of $\iota^* dx, \iota^* dy, \iota^* dz$, and $\iota^* dw$. For many purposes, the extra baggage of this pullback ι^* is unnecessarily cumbersome. However, there are a few places where its absence can lead to crucial errors. For example, the 2-form $dx \wedge dy$ on $\mathbb{R}^4$ never vanishes, while we will see that the 2-form $\iota^*(dx \wedge dy)$ on S^3 vanishes whenever $x = \pm 1$ or $y = \pm 1$ (this will come out of the computation below). For this reason, I will keep track of ι throughout this proof.

Proof of (1). Given the conversation of the previous paragraph, the goal is to show

$$*_{S^3}(\iota^*(dx \wedge dy)) = \sqrt{\frac{(1 - x^2)(1 - y^2)}{z^2 + w^2}}\, \iota^*(zdw - wdz),$$

where $\iota : S^3 \to \mathbb{R}^4$ is the inclusion. It suffices to assume we are working at a point $p = (x, y, z, w)$ where $\iota^*(dx \wedge dy)$ does not vanish; this implies that $\iota^* dx$ and $\iota^* dy$ are linearly independent.

Write $\pi : \mathbb{R}^4 \to T_pS^3$ for the orthogonal projection onto the tangent space. Then the pullback $\pi^* : T_p^*S^3 \to (\mathbb{R}^4)^*$ is an isometric embedding with image the 3-plane having normal vector

$$\nu := xdx + ydy + zdz + wdw.$$

Then $P^* := (\iota \circ \pi)^* : (\mathbb{R}^4)^* \to (\mathbb{R}^4)^*$ is the orthogonal projection operator with image $T_p^*S^3$; that is,

$$P^* \circ P^* = P^*, \qquad P^*|_{\pi^*(T_p^*S^3)} = \mathrm{Id}_{\pi^*(T_p^*S^3)}.$$

We will see that this is a useful operator to have around.

Consider the covector

$$\beta' := *(\nu \wedge P^*dx \wedge P^*dy)$$

where the Hodge star is on $\mathbb{R}^4$. This is normal to ν, P^*dx and P^*dy. Since it is normal to ν, it follows that β' lies in $\pi^*T_p^*S^3$. The map π^* is injective, so we can therefore write β' as

$$\beta' = \pi^*\beta$$

for a unique $\beta \in T_p^*S^3$. Now let's use the fact that $\beta' = \pi^*\beta$ is normal to $P^*dx = \pi^*(\iota^*dx)$ and $P^*dy = \pi^*(\iota^*dy)$. This observation and the fact that π^* is an isometric embedding combine to imply that β is orthogonal to ι^*dx and ι^*dy. Likewise, the covector $*_{S^3}(\iota^*(dx \wedge dy))$ is orthogonal, in $T_p^*S^3$, to ι^*dx and ι^*dy. Since $T_p^*S^3$ is 3-dimensional, it follows that β and $*_{S^3}(\iota^*(dx \wedge dy))$ are colinear; thus, there is some scalar $c \in \mathbb{R}$ so that

$$*_{S^3}(\iota^*(dx \wedge dy)) = c\beta.$$

We can use the identity (2) to compute c:

$$\begin{aligned}
c\nu \wedge d\mathrm{vol}_{S^3} &= cd\mathrm{vol}_{\mathbb{R}^4} \\
\text{(by (2))} &= \tfrac{c}{|\beta'|^2}\nu \wedge P^*dx \wedge P^*dy \wedge \beta' \\
\text{(def. of } P^* \text{ and } \beta') &= \tfrac{c}{|\beta'|^2}\nu \wedge \pi^*\iota^*dx \wedge \pi^*\iota^*dy \wedge \pi^*\beta \\
&= \tfrac{1}{|\beta'|^2}\nu \wedge \pi^*\left(\iota^*(dx \wedge dy) \wedge c\beta\right) \\
\text{(def. of } c) &= \tfrac{1}{|\beta'|^2}\nu \wedge \pi^*\left(\iota^*(dx \wedge dy) \wedge *_{S^3}(\iota^*(dx \wedge dy))\right) \\
\text{(by (2))} &= \tfrac{|\iota^*(dx\wedge dy)|}{|\beta'|^2}\nu \wedge d\mathrm{vol}_{S^3}.
\end{aligned}$$

This shows

$$c = \frac{|\iota^*(dx \wedge dy)|}{|\beta'|}. \tag{3}$$

Thus, to compute $*_{S^3}(\iota^*(dx \wedge dy))$, we need to compute $\beta' = *(\nu \wedge P^*dx \wedge P^*dy)$. We begin by computing P^*dx and P^*dy. Recall that P^* is the orthogonal projection operator onto the 3-plane with normal vector ν. It follows that P^* is given by the usual vector calculus formula for projection:

$$P^*\alpha = \alpha - \langle \alpha, \nu \rangle \nu.$$

(Note that ν is a unit vector since $p = (x, y, z, w) \in S^3$.) Thus

$$\begin{array}{rcl} P^*dx & = & (1-x^2)dx - xydy - xzdz - xwdw \\ P^*dy & = & -xydx + (1-y^2)dy - yzdz - ywdw. \end{array} \tag{4}$$

Taking the wedge, we find

$$\begin{array}{rcl} P^*dx \wedge P^*dy & = & (1-x^2-y^2)dx \wedge dy - yzdx \wedge dz - ywdx \wedge dw \\ & & +xzdy \wedge dz + xwdy \wedge dw \end{array}$$

and so

$$\nu \wedge P^*dx \wedge P^*dy \quad = \quad wdx \wedge dy \wedge dw + zdx \wedge dy \wedge dz$$

which gives

$$\beta' = *(\nu \wedge P^*dx \wedge P^*dy) = zdw - wdz.$$

From this we can immediately read that

$$|\beta'| = \sqrt{z^2 + w^2}$$

which is the denominator on the right of (3). The formula for β' also allows us to compute β. Indeed, it follows from the construction above that $zdw - wdz$ is normal to ν, and so $zdw - wdz \in \pi^*(T_p^*S^3)$ lies in the space on which $P^* = \pi^*\iota^*$ acts as the identity. Thus

$$\pi^*\beta = \beta' = zdw - wdz = \pi^*\iota^*(zdw - wdz).$$

Since π^* is injective, this implies

$$\beta = \iota^*(zdw - wdz).$$

To finish the computation, we return to our formula (3) for c; we need to compute the numerator $|\iota^*(dx \wedge dy)|$. The operator π^* is an isometric embedding, so we have

$$|\iota^*(dx \wedge dy)| = |\pi^*\iota^*(dx \wedge dy)| = |P^*dx \wedge P^*dy|.$$

Since dx and dy are orthogonal, and P^* is an orthogonal projection operator, it follows that P^*dx and P^*dy are orthogonal. This gives

$$|\iota^*(dx \wedge dy)| = |P^*dx||P^*dy|.$$

Using the formulas in (4), we find

$$|P^*dx| = \sqrt{1-x^2}, \qquad |P^*dy| = \sqrt{1-y^2}.$$

Putting this all together, our desired formula follows:

$$
\begin{aligned}
*_{S^3}(\iota^*(dx \wedge dy)) &= c\beta \\
&= \frac{|\iota^*(dx \wedge dy)|}{|\beta'|}\iota^*(zdw - wdz) \\
&= \sqrt{\frac{(1-x^2)(1-y^2)}{z^2+w^2}}\iota^*(zdw - wdz).
\end{aligned}
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