Which real vector bundles over S^4 admit complex structures?

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Suppose $R \to S^4$ is a real vector bundle over S^4 of rank r. The main question I want to dive into here is the following: When is R the underlying real vector bundle $E_{\mathbb{R}}$ of a complex vector bundle $E \to S^4$? Obviously a necessary condition is that the rank r needs to be even. Is this also a sufficient condition? I will show that the answer is "yes", unless $r = 4$, where it is more complicated. The main thing I like about this question is that in seeking out its answer we get to see some fun machinery. It also shows that classification question for vector bundles can be surprisingly sensitive to rank, even for bundles over the 4-sphere.

Let's start by classifying the real vector bundles on S^4. Arguably the easiest way to do this on spheres is via the clutching function construction. In our case, this construction shows that real rank-r vector bundles on S^4 are classified by

$$\pi_3(\mathrm{SO}(r)) = \pi_3(\mathrm{Spin}(r)) \cong \begin{cases} 0 & \text{if } r \leq 2 \\ \mathbb{Z} & \text{if } r = 3 \\ \mathbb{Z}^2 & \text{if } r = 4 \\ \mathbb{Z} & \text{if } r \geq 5 \end{cases} \tag{1}$$

For those unfamiliar with this construction, it goes as follows: Since $H^1(S^4) = 0$, any real vector bundle $R \to S^4$ is automatically orientable. This means we can trivialize R in a way that all transition maps take values in $\mathrm{SO}(r)$. Since the upper- and lower-hemispheres of S^4 are contractible, we can trivialize R over each hemisphere to get a single transition map $S^3 \to \mathrm{SO}(r)$, where the S^3 appearing here should be viewed as the equator where the upper and lower hemispheres intersect. This transition map is what is often referred to as the "clutching function". The key feature is that this clutching function construction produces a one-to-one correspondence between real rank-r vector bundles over S^4 and homotopy class of maps from S^3 into $\mathrm{SO}(r)$. It is in this sense that (1) classifies rank-r real vector bundles over S^4. I should also mention that the isomorphism in (1) is only canonical up to a sign. This sign ambiguity comes from the transition map construction: do you go from the boundary of the upper hemisphere to the lower, or vice versa? It doesn't really matter, but we will pin down a convention shortly.

The classification in (1) makes the answer to our question relatively straightforward...for the most part. I will start by analyzing the easiest case $r \leq 2$, then the "typical case" $r \geq 5$ and finally the $r = 4$ case. In something of an epilogue I will wrap up some residual questions pertaining to the $r = 3$ case, even though it isn't directly relevant to the initial question from the introduction. Interestingly, the $r = 3$ case has some hidden surprises not appearing in the $r \geq 5$ case, even though $\mathbb{Z}$ classifies bundles in both cases.

1 Rank ≤ 2

If $r \leq 2$, and R has rank r, then $R \cong S^4 \times \mathbb{R}^r$ is trivializable. Thus R admits a complex structure if and only if $\mathbb{R}^r$ admits a complex structure, and this is the case if and only if r is even (still assuming $r \leq 2$).

2 Rank ≥ 5

In this case our real vector bundles are classified by an element of $\pi_3(\mathrm{SO}(r)) \cong \mathbb{Z}$. This integer is roughly the Pontrjagin class $p_1(R) \in H^4(S^4) \cong \mathbb{Z}$. The reasons I say "roughly" are two-fold. First, there is this matter of this sign ambiguity I mentioned above, but we could establish some convention to pin that down (I will do this shortly). Second, it turns out that the Pontrjagin class $p_1(R)$ is always *even* (for all ranks). To see this, recall that $p_1(R) = -c_2(R \otimes \mathbb{C})$ and that $c(E) \equiv w(E_{\mathbb{R}})$ mod 2 (see Milnor–Stasheff Exercise 14-B). Now use the Whitney product formula $w(R \oplus R') = w(R)w(R')$ and $(R \otimes \mathbb{C})_{\mathbb{R}} \cong R \oplus R$ to get

$$\begin{aligned} p_1(R) &\equiv -c_2(R \otimes \mathbb{C}) \bmod 2 \\ &\equiv w_4(R \oplus R) \bmod 2 \\ &\equiv w_2(R)^2 \bmod 2. \end{aligned}$$

Since $w_2(R) \in H^2(S^4, \mathbb{Z}_2) = 0$, this implies that $p_1(R)$ is even, as claimed.

Conversely, I will show the following.

Lemma 2.1. *For any rank $r \geq 4$ and any even number $2n$, there is a real vector bundle R of rank r and with $p_1(R) = -2n$.*

The reason for the negative sign in the statement of the lemma is to undo the negative sign in $p_1(R) = -c_2(R \otimes \mathbb{C})$ (I prefer my conventions to work out well for the Chern classes, with the Pontrjagin classes getting the scraps; sorry Pontrjagin). This lemma will allow me to pin down the aforementioned problem with the sign ambiguity (for $r \geq 5$): I want the isomorphism (1) to have the sign such that a bundle R with clutching function function $f : S^3 \to \mathrm{SO}(k)$ sends $[f] \in \pi_3(\mathrm{SO}(k))$ to the integer $-\frac{1}{2}p_1(R) \in \mathbb{Z}$ under (1).

To prove the lemma, let's construct some bundles: Consider the Hopf fibration $S^7 \to S^4$. This is a principal SU(2)-bundle with $c_2 = 1$. It follows that the associated complex vector bundle $E_1 := S^7 \times_{\mathrm{SU}(2)} \mathbb{C}^2$ has complex rank 2 and $c_2(E_1) = 1$. Let n be any integer and fix a map $f_n : S^4 \to S^4$ of degree n. Then $E_n := f_n^* E_1$ is a vector bundle over S^4 with complex rank 2 and $c_2(E_n) = n$. It follows that the underlying real vector bundle $R_n := (E_n)_\mathbb{R}$ has real rank 4 and $p_1(R_n) = -2n$. Since the Pontrjagin classes are stable invariants, it follows that for each $r \geq 4$ the bundle $R_n \times \mathbb{R}^{r-4}$ has real rank r and $p_1 = -2n$. This proves the lemma.

Along the way to proving Lemma 2.1, we have mostly answered our initial question in this case: By construction, the bundle $R_n = (E_n)_\mathbb{R}$ admits a complex structure. Thus, $R_n \times \mathbb{R}^{r-4}$ admits a complex structure if and only if $\mathbb{R}^{r-4}$ does, and this is the case if and only if r is even. Because of our classification in (1), this proves the following.

Theorem 2.2. *Suppose $R \to S^4$ is a real vector bundle of rank not equal to 4. Then R is the underlying real vector bundle of a complex vector bundle if and only if the rank is even.*

Let's linger on this $k \geq 5$ case a bit longer. Here I will take the principal bundle perspective, by passing to the frame bundle and realizing R as a principal $\mathrm{SO}(r)$-bundle. Let $G_{\mathrm{SO}(3)} \subseteq \mathrm{SO}(r)$ be the subgroups $\mathrm{diag}(A, 1, \ldots, 1)$ where $A \in \mathrm{SO}(3)$. Then the map $\mathrm{SU}(2) \to \mathrm{SO}(3) \to \mathrm{SO}(r)$ given by sending $x \in \mathrm{SU}(2)$ to $\mathrm{diag}(\mathrm{Ad}_x, 1, \ldots, 1)$ generates $\pi_3(\mathrm{SO}(r))$. What this implies is that the $\mathrm{SO}(r)$-bundle R has structure group reducible to SU(2) (e.g., take the clutching function for R and homotope it so it has image in G). That is, there is a principal SU(2)-bundle $P \to S^4$ with

$$R \cong P \times_{\mathrm{SU}(2)} \mathrm{SO}(r)$$

where the group homomorphism $\mathrm{SU}(2) \hookrightarrow \mathrm{SO}(r)$ is as above. This is a principal bundle version of the construction of $R_{n,r}$ used in the proof of Theorem 2.2.

3 Rank 4

Our proof of Theorem 2.2 breaks down for $r = 4$ only because $\pi_3(\mathrm{SO}(4))$ is $\mathbb{Z}^2$ and not $\mathbb{Z}$, so the Pontrjagin class (a single integer) couldn't possibly capture the richness of bundles in this setting. Interestingly, there is another characteristic class floating around, and this is the Euler class $e(R)$. In general, the Euler class lives in $H^r(S^4)$ and it is only when $r = 4$ that the Euler class has a chance of being non-zero. This makes for an interesting dynamic: The Pontrjagin class fails to classify rank 4 bundles, but it is only for rank 4 bundles that the Euler class produces a useful invariant. As you might have guessed, these two characteristic classes taken together classify rank-4 bundles over S^4:

Theorem 3.1. *Suppose $R, R' \to S^4$ are real vector bundles of rank 4. Then $R \cong R'$ if and only if $p_1(R) = p_1(R')$ and $e(R) = e(R')$.*

I will give something of a circuitous proof of this, extracted from Milnor's lovely little paper [2]. Let's go back to the clutching function construction in (1). The spin group $\mathrm{Spin}(4) \cong S^3 \times S^3$ is a product of spheres. Since Spin(4) is a double cover of SO(4), we have

$$\pi_3(\mathrm{SO}(4)) = \pi_3(\mathrm{Spin}(4)) \cong \pi_3(S^3) \times \pi_3(S^3).$$

The isomorphism $\pi_3(S^3) \cong \mathbb{Z}$ gives (1). Given $(h, j) \in \mathbb{Z}^2$ my present goal is to create a real rank-4 vector bundle $R_{h,j} \to S^4$ with clutching function given by $(h, j) \in \mathbb{Z}^2$ under (1). To do this, view $S^3 \subseteq \mathbb{H}$ as the group of unit quaternions. Note that if $u \in S^3$, then the map $v \mapsto u^h v u^j$ is linear on the real vector space $\mathbb{H} \cong \mathbb{R}^4$. Thus, we have a map

$$f_{h,j} : S^3 \longrightarrow \mathrm{SO}(4), \qquad u \longmapsto (v \mapsto u^h v u^j).$$

(I'm using Milnor's notation, for those who choose to look up his paper, which I highly recommend.) The homotopy class of $f_{h,j}$ produces an element of the group $\pi_3(\mathrm{SO}(4)) \cong \mathbb{Z}^2$ and one can check that this corresponds to $(h, j) \in \mathbb{Z}^2$, up to a sign depending on our transition function choices referenced above. (If you want to check this, it is predicated on the observation that the map $u \mapsto u^h$ on S^3 has degree h.) By using this function $f_{h,j}$ in the clutching construction, then we can produce a bundle $R_{h,j}$ with clutching function $f_{h,j}$; this has the desired property. I'm going to skip this part, but Milnor [2] also shows that

$$p_1(R_{h,j}) = \pm 2(h - j), \qquad e(R_{h,j}) = h + j \tag{2}$$

for some sign $\pm$. This sign ambiguity is the $r = 4$ version of the same one discussed above and so, by possibly redefining our clutching function, we can assume $\pm = -$ in (2). The reason for this sign choice is the same as above: I want the conventions arranged so that the appropriate second Chern number has no minus sign.

Since the function $(h, j) \mapsto (2(j - h), h + j)$ is injective, Theorem 3.1 follows from our classification (1).

We have made significant progress in our exploration of rank-4 real vector bundles over S^4. However, our initial question still remains in this case: Given a rank-4 real vector bundle $R \to S^4$, when does it admit a complex structure? The following example shows that this may not be as straight-forward as a parity question for the Euler class.

Example 3.2. *The tangent bundle TS^4 has $p_1(TS^4) = 0$ and $e(TS^4) = 2$, but has no complex structure (which would be an* almost *complex structure on S^4). Let's take these claims one at a time. Since p_1 is stable, we have $p_1(TS^4) = p_1(TS^4 \times \mathbb{R})$. The normal bundle to S^4 is trivializable, so $TS^4 \times \mathbb{R} \cong (T\mathbb{R}^5)|_{S^4}$ is the restriction to S^4 of the tangent bundle to $\mathbb{R}^5$. Clearly $T\mathbb{R}^5$ is trivializable, so its restriction to S^4 is as well. This implies $p_1(TS^4) = 0$.*

The claim about the Euler class is one of those fun Euler characteristic games: The identification $H^4(S^4) \cong \mathbb{Z}$ is given by pairing with the fundamental class $[S^4]$. Pairing with the fundamental class of S^4 we have the Euler characteristic

$$e(TS^4)[S^4] = \chi(S^4) = 2.$$

The really fun part is the complex structure bit. Suppose $TS^4 = E_{\mathbb{R}}$ for some complex vector bundle E, which is necessarily of complex rank 2. Then

$$p_1(TS^4) = -c_2(TS^4 \otimes \mathbb{C}) = -c_2(E \oplus E^*) = -2c_2(E) = -2e(E_{\mathbb{R}}) = -4. \quad (3)$$

This is a contradiction since $p_1(TS^4) = 0$.

To figure this complex structure thing out, let's go in the other direction: Suppose $E \to S^4$ is a complex rank-2 bundle. The same type of argument used for (3) shows

$$p_1(E_{\mathbb{R}}) = -2e(E_{\mathbb{R}}).$$

This gives us another necessary condition. Is *it* sufficient? The answer here is *yes.*

Theorem 3.3. *Suppose $R \to S^4$ is a real vector bundle of rank 4. Then R admits a complex structure if and only if $p_1(R) = -2e(R)$.*

We have already proven one direction, so let's assume $p_1(R) = -2e(R)$. It suffices to work under the assumption that $R = R_{h,j}$ is one of our representative bundles. Then we are assuming

$$2(j - h) = -2(h + j).$$

This implies $j = 0$. Our bundle $R_{h,0}$ is obtained from the clutching function $f_{h,0}$ that sends $u \in S^3$ to the map $v \mapsto \big(f_{h,0}(u)\big)(v) := u^h v$. Let $i = \sqrt{-1} \in \mathbb{H}$, and notice that $f_{h,0}(u)$ commutes with right multiplication by i for all u:

$$\big(f_{h,0}(u)\big)(vi) = \Big(\big(f_{h,0}(u)\big)(v)\Big)i.$$

This implies that right multiplication by i on each trivialization used to define $R_{h,0}$ descends to give a well-defined action of i on $R_{h,0}$, thus giving it a complex structure. This proves the theorem.

Recall that there was a choice of convention of sign $\pm$ in (2) and we had gone with $\pm = -$. If we had gone with $\pm = +$, then this would force $h = 0$ (not j) and *left* multiplication by i would descend to an action on $R_{0,j}$. Either way, the statement of Theorem 3.3 would continue to hold as stated, regardless of our sign conventions, as it should.

4 Rank 3 — Divisibility of the Pontrjagin class

We have seen that the Pontrjagin class $p_1(R)$ of a real vector bundle on S^4 is always even. When the rank is at most 2 this class is zero, and Lemma 2.1 shows that for rank ≥ 4 the Pontrjagin class takes on all even numbers. What about rank 3? Here is a challenge for the reader before reading on: Try to find a rank-3 vector bundle on S^4 with Pontrjagin class 2 (or -2, I don't care).

[Spoiler Alert!]

Here is a fun fact: If R is a rank-3 real vector bundle over a closed, connected, oriented 4-manifold, then

$$\mathfrak{P}(w_2(R)) \equiv p_1(R) \qquad \bmod 4,$$

where $\mathfrak{P}$ is the Pontrjagin square. This formula seems to go back to Wu in the 50's but it is quoted in [1, (2.1.36)], among many other places. The usefulness of this formula for us is that, if $H^2 = 0$ (as is the case for us), then all rank-3 vector bundles R have $p_1(R)$ divisible by 4. What the hell? This is new! Is there some other weird necessary condition lurking for rank-3 vector bundles on S^4, or is this the only remaining one?

Well, as it turns out, this is it.

Theorem 4.1. *For every integer n, there is a real rank-3 vector bundle $R \to S^4$ with $p_1(R) = -4n$.*

Here's a slick way to do it, which I am pulling right out of [1, Ch. 2]: Above we constructed a complex vector bundle $E_n \to S^4$ of rank 4 and $c_2(E_n) = n$. Consider the endomorphism bundle $R_{n,3} := \mathfrak{su}(E_n)$ consisting of skew-symmetric, trace zero endomorphisms of E_n. This has real rank 3 (since $\mathfrak{su}(2) \cong \mathfrak{so}(3) \cong \mathbb{R}^3$), so we are in the ballpark by at least getting the rank right. Moreover, there is an isomorphism between $R_{n,3}$ and the *symmetric*, trace zero endomorphisms of E_n; this is given by multiplication by i, so the latter space is $iR_{n,3}$. The space of central endomorphisms of E_n is a trivial bundle of complex rank 1. Thus, there is an isomorphism

$$R_{n,3} \oplus iR_{n,3} \oplus \mathbb{R}^2 \cong \mathrm{End}(E_n)_{\mathbb{R}}$$

of real vector bundles. Since $\mathrm{End}(E_n) = E_n \otimes E_n^*$, this gives

$$2p_1(R_{n,3}) = p_1(\mathrm{End}(E_n)_{\mathbb{R}}) = -c_2(\mathrm{End}(E_n)) = -c_2(E_n \otimes E_n^*).$$

To compute the right-hand side, let's use the Chern character ch since it satisfies the multiplicative formula $\mathrm{ch}(E \otimes F) = \mathrm{ch}(E)\mathrm{ch}(F)$. Note also that

$$\mathrm{ch}(E) = k - c_2(E)$$

for complex rank-k vector bundles $E \to S^4$. This gives

$$4 - c_2(E \otimes E^*) = \mathrm{ch}(E_n \otimes E_n^*) = \mathrm{ch}(E_n)\mathrm{ch}(E_n^*) = (2 - c_2(E_n))^2 = 4 - 4c_2(E_n).$$

Thus $c_2(E \otimes E^*) = 4c_2(E_n)$. We therefore conclude that

$$p_1(R_{n,3}) = -4c_2(E_n) = -4n.$$

This proves the theorem.

References

[1] S. Donaldson, P.B. Kronheimer. *The Geometry of Four-Manifolds*, Clarendon Press, Oxford 1990.

[2] J. Milnor. On manifolds homeomorphic to the 7-sphere. *The Annals of Mathematics*, Second Series, Vol. 64, No. 2 (Sep., 1956), pp. 399–405

[3] J. Milnor, J. Stasheff. Lectures on characteristic classes, *Ann. of Math.* Studies 76 (1974).