The quilted Atiyah-Floer conjecture and the Yang-Mills heat flow

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Assume this is chosen so that the induced bundle on Y admits no reducible flat connections.

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Fix $a \in \mathcal{A}(Y)$, and let $\mathcal{A}(X; a)$ denote the space of $W^{1,2}$ connections on X that are asymptotic to a.

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The global minimizers of $\mathcal{YM}$ on $\mathcal{A}(X; a)$ are the *anti-self dual (ASD) connections:*

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The *Yang-Mills heat flow* is the negative gradient flow of $\mathcal{YM} : \mathcal{A}(X; a) \to \mathbb{R}$

$$\partial_\tau A = -d_A^* F_A, \qquad A(\tau) = A_0$$

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Resolution: Introduce a perturbation to make it Fredholm.

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H can be chosen so that this satisfies the Bianchi identity $d_{A,H} F_{A,H} = 0$, where $d_{A,H}$ is a perturbed version of d_A.

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Moreover, H can be chosen so all h-flat connections are non-degenerate (and so we have a Fredholm problem).

Short-time existence and uniqueness

Theorem (99% Struwe '94, 1% D. '14)

There is a suitable class of perturbations $\mathcal{H}$ so that if $H \in \mathcal{H}$ is such that all h-flat connections are non-degenerate, then short-time existence and uniqueness holds for the perturbed Yang-Mills heat flow.

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Check that the perturbation does not mess up the estimates.

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(Roughly speaking, this means the absolute minima of $\mathcal{YM}_H$ are *Morse* critical points, when taken mod gauge.)

Corollary (D. '14)

Assume $G = \mathrm{SO}(3)$ *and* H *is regular. Suppose* a *is an h-flat connection with* $\mathrm{Ind}(d^+_{A,H} \oplus d^*_{A,H}) < 8$, *for* $A \in \mathcal{A}(X; a)$.

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- *The $A(\tau)$ converge exponentially to a unique H-ASD connection $A_\infty \in \mathcal{A}(X; a)$.*
- *There is a constant C so $\|F^+_{A(\tau),H}\|_{L^2} \leq C\|d^*_{A(\tau),H} F_{A(\tau),H}\|_{L^2}$ for all $\tau \in [0, \infty]$. Moreover,*

$$\|A_0 - A_\infty\|_{L^2} \leq (1 - e^{-2/C^2})^{-1} \|F^+_{A_0}\|_{L^2}.$$

An application

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$P \to Y$, an $\mathrm{SO}(3)$-bundle with $w_2(P) \in H^2(Y, \mathbb{Z}_2)$ in the image of a generator of $H^2(Y, \mathbb{Z})/\mathrm{torsion}$

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$X = \mathbb{R} \times Y$, with cylindrical metric and induced bundle.

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The counts depend only on Y and the topological type of P.

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Theorem (D. '14)

There are g, H so that the H-perturbed Yang-Mills heat flow induces an injection

$$\mathcal{M}_{\mathrm{holo}}(g,H) \to \mathcal{M}_{\mathrm{ASD}}(g,H).$$

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Theorem

Let $\eta(g_n) > 0$ be the constant from the perturbed Yang-Mills heat flow. Then $\inf_n \eta(g_n) > 0$.

Sketch of proof (cont'd)

Corollary

The for all n sufficiently large, the perturbed Yang-Mills heat flow restricts to define a map

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Injectivity: Show the constant $C = C(g_n)$ (from the heat flow theorem) can be taken independent of n. Then use the estimate

$$\|A_0 - A_\infty\|_{L^2} \leq (1 - e^{-2/C^2})^{-1} \|F^+_{A_0,H}\|_{L^2}$$

Work in progress/future directions

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Use the harmonic map flow? (On a surface with boundary and cylindrical/strip-like ends.)

Thank you for your attention!