A proof that TS^2 has $c_1 = 2$ (with some pictures!)

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The goal of this note is to show that the tangent bundle $TS^2 \to S^2$ has first Chern number equal to 2 using a geometric approach. The first step reduces the computation to that of computing the degree of a transition map (the clutching function), and the second step computes the degree via parallel transport.

Think of S^2 as embedded in $\mathbb{R}^3$ in the usual way, and write $\mathbb{R}^2 = \mathbb{R}^2 \oplus 0 \subseteq \mathbb{R}^3$ for the xy-plane. Denote the equator by $S^1_{eq} = S^2 \cap (\mathbb{R}^2 \oplus 0)$, and let $n \in S^2$ and $s \in S^2$ be the North and South Poles, respectively. Let $H_n \subseteq S^2$ be a closed neighborhood of the Northern Hemisphere, and H_s a closed neighborhood of the Southern. Take these to be closed subsets diffeomorphic to a disk, so the intersection

$$H_n \cap H_s \cong S^1_{eq} \times [-\epsilon, \epsilon] \tag{1}$$

is a collar neighborhood of the equator. Orient this equatorial circle to have the usual orientation when viewed as a subset of the xy-plane; this makes the identification (1) orientation-preserving.

Suppose $P \to S^2$ is a principal S^1-bundle; in the end this will be the unit tangent bundle $S(TS^2) \subseteq TS^2$. For $x \in S^2$, write P_x for the fiber in P over x. Fix trivializations

$$\phi_n : P|_{H_n} \xrightarrow{\cong} H_n \times S^1, \qquad \phi_s : P|_{H_s} \xrightarrow{\cong} H_s \times S^1.$$

These will be refined in our case $P = S(TS^2)$ shortly. From any two such trivializations, we obtain a transition function

$$\tau : S^1_{eq} \longrightarrow S^1$$

by using ϕ_s^{-1} first, then ϕ_n:

$$\phi_n \circ \phi_s^{-1}(b, g) = (b, \tau(b)g) \tag{2}$$

for $b \in S^1_{eq}$ and $g \in S^1$. It is well-known that the isomorphism class of P is uniquely determined by the homotopy class of τ; this is the clutching function construction (τ is the *clutching function*). More specifically, the map sending P to the homotopy class of τ descends to give a bijection

$$\frac{\{\text{principal } S^1\text{-bundles over } S^2\}}{\text{isomorphism}} \cong [S^1_{eq}, S^1] \cong \mathbb{Z}. \tag{3}$$

The integer obtained this way can be viewed either as the first Chern number

$$c_1(P)[S^2] \in \mathbb{Z}$$

of the bundle, of as the degree $\deg(\tau) \in \mathbb{Z}$ of the transition map $\tau : S^1_{eq} \to S^1$. It doesn't really matter which because these two integers are equal:

Theorem 0.1.

$$c_1(P)[S^2] = \deg(\tau).$$

Proof. I will use the Chern–Weil formula

$$c_1(P)[S^2] = \frac{i}{2\pi} \int_{S^2} F_A$$

where A is any connection on P and F_A is its curvature. To create a convenient connection for computation, write A_{triv} for the trivial connection and set

$$A_n := \phi_n^* A_{\text{triv}}, \qquad A_s := \phi_s^* A_{\text{triv}}$$

which are connections on $P|_{H_n}$ and $P|_{H_s}$, respectively. Recall the collar neighborhood $N := [-\epsilon, \epsilon] \times S^1_{eq}$ of $S^1_{eq} \subseteq S^2$ and define a connection A on P as follows:

- A should equal A_n on $H_n \backslash N$;
- A should equal A_s on $H_s \backslash N$;
- on $N = [0, 1] \times S^1_{eq}$, the connection A should interpolate between A_s and A_n by

  $$A|_{[0,1]\times S^1_{eq}} = A_s + \frac{1}{2\epsilon}(t + \epsilon)(A_t - A_s)$$

  where t is the parameter on $[0, 1]$.

Then $F_A = 0$ on the complement of N, since A equals the pullback of a trivial (flat) connection there. This implies

$$c_1(P)[S^2] = \frac{i}{2\pi} \int_N F_A.$$

On N use ϕ_n to trivialize the bundle P. Then A pulls back under $\phi_s^{-1} : N \times S^1 \to P|_N$ to the connection

$$A_{\text{triv}} + \frac{1}{2\epsilon}(t + \epsilon)(\tau^* A_{\text{triv}} - A_{\text{triv}}).$$

Now, the formula $\tau^* A_{\text{triv}} = A_{\text{triv}} + d\tau$ gives

$$(\phi_s^{-1})^* A\big|_N = A_{\text{triv}} + \frac{1}{2\epsilon}(t + \epsilon)\tau^{-1} d\tau.$$

Then

$$F_A\big|_N = d\Big((\phi_s^{-1})^* A\big|_N\Big) = d\Big(\frac{1}{2\epsilon}(t+\epsilon)\tau^{-1}d\tau\Big).$$

is exact on N and so Stokes' theorem gives

$$\begin{aligned} c_1(P)[S^2] &= \tfrac{i}{2\pi}\int_N F_A \\ &= \tfrac{i}{2\pi}\int_{\partial N} \frac{1}{2\epsilon}(t+\epsilon)\tau^{-1}d\tau. \end{aligned}$$

Since $N \cong S^1_{eq} \times [-\epsilon, \epsilon]$ is orientation-preserving, it follows that

$$\partial N \cong \Big(S^1_{eq} \times \{-\epsilon\}\Big) \sqcup \Big(\overline{S^1_{eq}} \times \{\epsilon\}\Big)$$

where the overline in $\overline{S^1_{eq}}$ means the circle S^1_{eq} equipped with the opposite orientation. The integrand vanishes on $S^1_{eq} \times \{-\epsilon\}$ and picks up a minus sign on $\overline{S^1_{eq}} \times \{\epsilon\}$ due to the orientation-reversal to give

$$\begin{aligned} c_1(P)[S^2] &= \tfrac{i}{2\pi}\int_{\overline{S^1_{eq}}} \tau^{-1}d\tau \\ &= -\tfrac{i}{2\pi}\int_{S^1_{eq}} \tau^{-1}d\tau. \end{aligned}$$

The degree of $\tau : S^1_{eq} \to S^1$ is its winding number. That is, using $e^{i\theta}$ to parametrize $S^1_{eq} \cong S^1$, we can write $\tau(e^{i\theta}) = e^{iw(\theta)}$ for some real-valued function $w : [0, 2\pi] \to \mathbb{R}$ satisfying $w(2\pi) = w(0) + 2\pi \deg(\tau)$. Then $\tau^{-1}d\tau = iw'(\theta)$ and so we can continue the above to get

$$\begin{aligned} c_1(P)[S^2] &= -\tfrac{i}{2\pi}\int_{S^1_{eq}} \tau^{-1}d\tau \\ &= \tfrac{1}{2\pi}\int_0^{2\pi} w'(\theta)d\theta \\ &= \deg(\tau). \end{aligned}$$

□

With this in hand, let's specialize to the case where $P = S(TS^2)$ is the unit tangent bundle to S^2. A nice thing about this is that I can be very explicit about defining the trivializations ϕ_n, ϕ_s. To pin these down, first fix $v_n \in P_n$ and $v_s \in P_s$ to be unit vectors parallel to the positive y-axis; that is, viewed as vectors in $\mathbb{R}^3$, v_n and v_s should be $(0, 1, 0)$ but rooted at n and s, respectively. The choices of v_n and v_s canonically induce S^1-equivariant maps $P_n \cong S^1$ and $P_s \cong S^1$ by sending v_n and v_s to $i \in S^1$. See Figure 1. Then define ϕ_n, ϕ_s by parallel translating unit tangent vectors along North-to-South meridians on S^2.

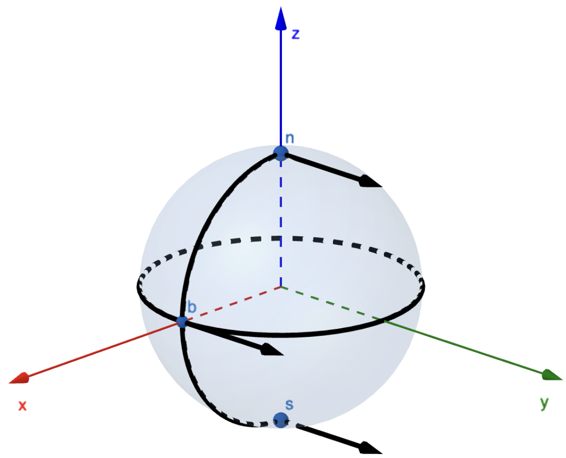

Figure 1: *Parallel transport of the vector v_s at the South Pole along the meridian through $b = (1, 0, 0)$ produces the vector v_n at the North Pole.*

To compute the transition function, let $b \in S^1_{eq} \subseteq H_n \cap H_s$ and $g \in S^1$. Write $PT^b_s : P_s \to P_b$ for parallel transport along the meridian from s to b. The definition of ϕ_s gives

$$\phi_s^{-1}(b, g) = PT^b_s(v_s g) = PT^b_s(v_s)g \in P_b$$

where concatenation with g means the S^1 action on P. A similar formula holds for ϕ_n:

$$\phi_n^{-1}(b, g) = PT^b_n(v_n g) = PT^b_n(v_n)g \in P_b.$$

By the defining formula (2) of $\tau(b) \in S^1$, we then have

$$PT^b_s(v_s) = \phi_s^{-1}(b, 1) = \phi_n^{-1}(b, \tau(b)) = PT^b_n(v_n)\tau(b).$$

That is, $\tau(b) \in S^1$ is the unique circle element taking $PT^b_n(v_n) \in P_b$ to $PT^b_s(v_s) \in P_b$. Equivalently, $\tau(b)$ is the unique circle element taking $v_n \in P_n$ to the vector $(PT^b_n)^{-1} \circ PT^b_s(v_s) \in P_n$. The operation $(PT^b_n)^{-1} \circ PT^b_s$ is parallel transport from the South to the North Pole, along the meridian passing through b. Thus, to determine $\tau(b)$, we parallel transport v_s through this meridian and compare its position relative to v_n.

We can get a feel for how this works for specific values of $b \in S^1_{eq}$. First, let's take $b = (1, 0, 0)$ to lie on the positive x-axis. Since v_n and v_s are both parallel to the y-axis, they are orthogonal to the meridian through b and remain so throughout the parallel translation along that line. As such we have $PT^b_s(v_s) = PT^b_n(v_n)$ by construction, and so $v_n = (PT^b_n)^{-1} \circ PT^b_s(v_s)$, which gives

$$\tau(1, 0, 0) = 1.$$

Now let's work out the case where $b = (0, 1, 0)$ lies on the positive y-axis. In this case, v_n and v_s are both parallel to the meridian through b, but $PT_s^b(v_s) = -PT_n^b(v_n)$. This gives

$$\tau(0, 1, 0) = -1.$$

See Figure 2.

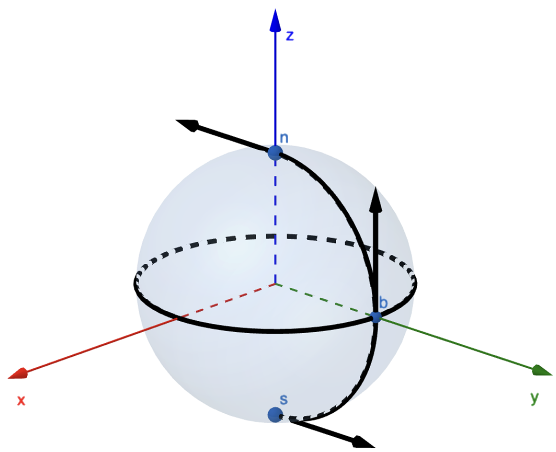

Figure 2: *Parallel transport of the vector v_s at the South Pole along the meridian through $b = (0, 1, 0)$ produces the vector $-v_n$ at the North Pole.*

In general, suppose $b = (\cos(\theta), \sin(\theta), 0)$. Note that v_s makes an angle of $\pi/2 - \theta$ relative to the meridian from s to n through b (oriented from s to n). Since parallel transport preserves angles, this implies that $PT_s^b(v_s)$ makes an angle of $\pi/2 - \theta$ relative to this same meridian (still oriented from s to n). Likewise, $PT_n^b(v_n)$ makes an angle of $\pi/2 - \theta$ relative to the meridian *oriented from n to s*. To compare, we should stick with the same orientation of this meridian for both: Orienting our meridian from s to n it follows that $PT_n^b(v_n)$ makes an angle of $-(\pi/2 - \theta) = \theta - \pi/2$ with this meridian. This requires an angle of $2\theta - \pi$ to get from $PT_s^b(v_s)$ to $PT_n^b(v_n)$, which implies

$$\tau(\cos(\theta), \sin(\theta), 0) = e^{(2\theta - \pi)i} = -e^{2\theta i}.$$

Identifying $S^1_{eq} = S^1$ via

$$(x, y, 0) \longmapsto x + iy,$$

the map τ is the squaring map $\tau(b) = -b^2$ (with a minus sign). This map has degree 2, which is what we were after (it is 2-1 and orientation-preserving).

Thanks to GeoGebra for the pictures!