Toroiding a prism:
When does this preserve volume?

David Duncan

One of my Calculus II students was trying to compute the volume of a solid torus in $\mathbb{R}^3$ obtained by rotating a circle of radius r around an axis a distance $R \geq r$ away. He tried to compute this by visualizing it as a cylinder with height $2\pi R$ and base a disk of radius r. Of course, these two are not isometric, even locally but, interestingly, they have the same volume. This note describes a way to see why (see Theorem 1.4), and explores some of the surrounding ideas.

1 Volumes of revolution

Let's work a little more broadly to start: Let B be a closed and bounded region in the plane $\mathbb{R}^2$. Given a positive real number h, we can create a *prism* $P_{B,h}$ with base B and height h: View $P_{B,h}$ as being a subset of $\mathbb{R}^3$, with $\mathbb{R}^2 \subseteq \mathbb{R}^3$ the xy-plane and the height of $P_{B,h}$ along the z-axis.

Next, let $a \subseteq \mathbb{R}^2$ be a line in the plane. We will always assume that B lies entirely on one side of this line. Let $T_{B,a} \subseteq \mathbb{R}^3$ be the volume of revolution obtained by rotating B about a. Call this the *toroid* associated to B and a.

With these definitions, my student's observation can be articulated as in the following example.

Example 1.1. *Suppose D is a disk in the right-half plane and R is the distance from the center of D to the y-axis. Then $P_{D,2\pi R}$ is a cylinder and $T_{D,y\text{-axis}}$ is a torus, and these have the same volume:*

$$\mathrm{Vol}(P_{D,2\pi R}) = \mathrm{Vol}(T_{D,y\text{-axis}}) = 2\pi^2 r^2 R.$$

Motivated by this, from now on I will assume that h is 2π times the distance from a to the center of mass (centroid) of B. Under such an assumption, the value of h is uniquely determined by a and B and, to emphasize this dependence, I will set

$$P_{B,a} := P_{B,h}.$$

In general, the volumes of $P_{B,a}$ and $T_{B,a}$ need not be equal, as most Calculus II students could detect. Here is a specific example that you can work out by hand, if you feel so inclined.

Example 1.2. *Suppose I is the isosceles triangle with vertices* $(1,4),(3,5),(3,3)$*, and a is the y-axis. Then $P_{I,y\text{-}axis}$ and $T_{I,y\text{-}axis}$ do not have the same volume.*

Interestingly, if we take the same triangle from the previous example, but use the x-axis instead of the y-axis, then we do recover the same volume!

Example 1.3. *Suppose I is as in Example 1.2. Then*

$$\mathrm{Vol}(P_{I,x\text{-}axis}) = \mathrm{Vol}(T_{I,x\text{-}axis}).$$

The following theorem sheds light on what is going on here and gives a framework for understanding my student's observation:

Theorem 1.4. *Suppose B is reflection-symmetric with symmetry line given by a_B, and let a be a second line. Assume B lies entirely on one side of a. If a is parallel to a_B, then the volumes*

$$\mathrm{Vol}(P_{B,a}) = \mathrm{Vol}(T_{B,a})$$

are equal.

Examples 1.1 and 1.3 are special cases, but Example 1.2 is not (indeed, it must not be).

Proof of Theorem 1.4. For each $x \in a$, define A_x to be the intersection of $T_{B,a}$ and the plane in $\mathbb{R}^3$ containing x and perpendicular to a; I will call A_x a *cross-section*. The volume of $T_{B,a}$ can be written as the integral

$$\mathrm{Vol}(T_{B,a}) = \int_a \mathrm{Area}(A_x)\, dx$$

of these cross-sections. Note that A_x is an annulus; let $r_1(x)$ and $r_2(x)$ be the inner radius and outer radius of A_x, respectively. Then the area of A_x is given by

$$\mathrm{Area}(A_x) = \pi r_2(x)^2 - \pi r_1(x)^2 = \pi(r_2(x) + r_1(x))(r_2(x) - r_1(x)). \quad (1)$$

Our symmetry assumption on B implies that

$$h = \pi(r_2(x) + r_1(x))$$

is independent of x. The difference $r_2(x) - r_1(x)$ is the length of the line segment given by the intersection of $B \subseteq \mathbb{R}^2$ and the line in $\mathbb{R}^2$ passing through x perpendicular to a. Since

$$\mathrm{Area}(B) = \int_a r_2(x) - r_1(x)\, dx,$$

we have

$$\begin{aligned} \mathrm{Vol}(T_{B,a}) &= \int_a \mathrm{Area}(A_x)\, dx \\ &= h \int_a r_2(x) - r_1(x)\, dx \\ &= h\mathrm{Area}(B), \end{aligned}$$

which is the volume of $T_{B,a}$. □

Exercise 1. *In the statement of Theorem 1.4 we assumed that B is reflection-symmetric about some line. Show that this is not a necessary condition: Show that there are B and a with*

$$\mathrm{Vol}(P_{B,a}) = \mathrm{Vol}(T_{B,a})$$

but where B is not reflection-symmetric about any line.

2 A 2-dimensional analogue

It is interesting to see what goes on one dimension lower. Let $C \subseteq \mathbb{R}^2$ be a rectifiable curve that is compact and not self-intersecting (e.g., a finite union of embedded circles and intervals). Write ℓ for the length of C. Let $R_{C,\epsilon}$ be the ϵ-neighborhood of C, where $\epsilon > 0$ is any number small enough so that this neighborhood is not self-intersecting (compactness at play here!). The first observation is the following.

Proposition 2.1. *The area of $R_{C,\epsilon}$ is $2\ell\epsilon$.*

Proof. First we prove the proposition under the assumption that C is smooth.

Let's make a Riemann sum: Partition C into arcs $i_1, \ldots, i_N$, each of length ℓ/N. Fix one of these arcs i_j and consider the two line segments in $R_{C,\epsilon}$ that intersect C perpendicularly at the endpoints of i_j (this is meaningful because C is smooth). The convex hull of these line segments is a trapezoid t_j. Then the set $t_1, \ldots, t_N$ of trapezoids nearly covers $R_{C,\epsilon}$. Adding the areas of the t_i is our Riemann sum. Since C is smooth, the area of $R_{C,\epsilon}$ equals the limit:

$$\mathrm{Area}(R_{C,\epsilon}) = \lim_{N\to\infty} \sum_{j=1}^{N} \mathrm{Area}(t_j).$$

On the other hand, the area of t_j is its height times the average of the lengths of the bases. This average is exactly the length of i_j, and this height is approximately 2ϵ (with the approximation getting better as N gets larger). This implies

$$\sum_{j=1}^{N} \mathrm{Area}(t_j) \approx 2\epsilon\ell.$$

Taking the limit in N proves the proposition.

If C is not smooth, then we can approximate it by a sequence $C_1, C_2, \ldots$ of curves that are smooth, each of which has length ℓ. Then we have

$$\lim_{i\to\infty} \mathrm{Area}(R_{C_i,\epsilon}) = \mathrm{Area}(R_{C,\epsilon}).$$

By the previous case, we have $\mathrm{Area}(R_{C_i,\epsilon}) = 2\ell\epsilon$, which finishes the proof. □

This proposition says that the area of $R_{C,\epsilon}$ is independent of how C curves: all that matters is how long C is and how much you thickened it up. You can think of this proposition as a curvy 2-dimensional version of Cavalieri's principle.

Example 2.2. *Consider the case where C is a circle in the plane of radius r. The proposition implies*

$$\operatorname{Area}(R_{C,\epsilon}) = 4\pi r\epsilon. \tag{2}$$

Incidentally, Example 2.2 gives another proof of the identity (1) used in the proof of Theorem 1.4: Suppose A is an annulus of inner radius r_1 and outer radius r_2. Then $A = R_{C,\epsilon}$ where $\epsilon = \frac{1}{2}(r_2 - r_1)$, and C is a circle with radius $r = \frac{1}{2}(r_2 + r_1)$. Then (1) reduces directly to (2). In this sense, we can view the 3-dimensional Theorem 1.4 as being a consequence of the 2-dimensional Proposition 2.1.

One final comment: At the heart of Proposition 2.1 is the geometric fact that the area of a trapezoid with a given height η and base lengths b_1, b_2 is the same as the area of a rectangle with height η and base $(b_1 + b_2)/2$. Interestingly, the corollary to Proposition 2.1 that is expressed in Example 2.2 gives something of a curvy version of this: It says that the area of an annulus of inner radius r_1 and outer radius r_2 equals the area of the rectangle with height $\pi(r_2 + r_1)$ and width $r_2 - r_1$. This latter statement that is the true 2-dimensional analogue of Theorem 1.4.