Natural connections on Lie groups

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The aim here is to discuss several ways of constructing connections on Lie group and manifolds closely related to Lie groups. These are all modifications of the Maurer–Cartan form ω, so I will review that first. Throughout, G is a Lie group with Lie algebra $\mathfrak{g}$. Depending on context, some of the conversation more naturally takes place on vector bundles (such as TG), while other bits more naturally take place on principal bundles. As such, I have included a review in the appendix of connections in each of these setting; I also use that space to establish some notation. Take a look there if you don't know what I mean by something. It might help. Though it might not.

The first few sections start off a bit slow, but I try to be careful about being explicit with conventions that have confused me at various times. The reader may find the examples in these sections fun, even if the broader discussion in these sections seems overly pedantic. In particular, I give a computation of the first Chern class of the Hopf fibration, which some readers may find value in reading (I kept getting the dang sign wrong, which motivated my aforementioned pedantry in the earlier sections). The construction of the last section is more novel, though I am sure plenty of experts have encountered constructions of its type before. It gives an infinite family of connections on products of Lie groups and gives some curvature calculations (the curvature is not always zero!).

1 The Maurer–Cartan forms

For $g \in G$, write

$$\begin{array}{rcl} L_g : G & \longrightarrow & G \\ h & \longmapsto & gh \end{array} \qquad\qquad \begin{array}{rcl} R_g : G & \longrightarrow & G \\ h & \longmapsto & hg \end{array}$$

for the left- and right-multiplication maps, respectively. The *left-invariant Maurer–Cartan form* is the 1-form $\omega_L \in \Omega^1(G, \mathfrak{g})$ defined at $v \in T_gG$ by

$$\omega_L(v) = (L_{g^{-1}})_* v \in T_eG = \mathfrak{g}.$$

Similarly, the *right-invariant Maurer–Cartan form* is the 1-form $\omega_R \in \Omega^1(G, \mathfrak{g})$ defined at $v \in T_gG$ by

$$\omega_R(v) = (R_{g^{-1}})_* v \in T_eG = \mathfrak{g}.$$

After the present section, I will focus mainly on ω_L, and so I will drop the "L" subscript and set:

$$\omega := \omega_L.$$

1.1 Some notation conversations

People often write the Maurer–Cartan forms as

$$\omega_L(v) = g^{-1}v, \qquad \omega_R(v) = vg^{-1}, \qquad \forall v \in T_gG. \tag{1}$$

This is a slight abuse of notation, but it is justified whenever G is a matrix group. In a bit more detail, assume $G \subseteq \mathrm{GL}(V)$ is a subgroup of the general linear group of some vector space V. Since $\mathrm{GL}(V) \subseteq \mathrm{End}(V)$ and $\mathrm{End}(V)$ is a vector space, it follows that G and its Lie algebra $\mathfrak{g}$ are subsets of $\mathrm{End}(V)$ as well:

$$G, \mathfrak{g} \subseteq \mathrm{End}(V).$$

Thus, if $g \in G$ and $\xi \in \mathfrak{g}$, then the concatenation $g\xi$ can be defined to be the multiplication in $\mathrm{End}(V)$ (which is composition of functions). The formulas (1) can be understood in this way.

Another way people often write the Maurer–Cartan forms is as

$$\omega_L = g^{-1}dg, \qquad \omega_R = (dg)g^{-1}. \tag{2}$$

First, as in (1), these formulas should be interpreted as taking place on the specific tangent space T_gG at g (and not necessarily on all of TG). Second, and also as in (1), the formulas (2) secretly assume that there is some way of "multiplying" elements of G and $\mathfrak{g}$, such as embedding G into a matrix group $\mathrm{GL}(V)$ as in the previous paragraph. What I think remains to be explained is what dg means in (2). To explain this, consider the identity map $G \to G$ given by sending $g \mapsto g$. As in calculus, write this simply as g (think of the function $f(x) = x$). We can view this function g as a function of the form $G \to \mathrm{End}(V)$. Thus, we can think of the identity function g as an element of the space $\Omega^0(G, \mathrm{End}(V))$ of $\mathrm{End}(V)$-valued 0-forms on G (see Appendix A.1 for details on this space of forms). Let $d : \Omega^0(G, \mathrm{End}(V)) \to \Omega^1(G, \mathrm{End}(V))$ be the trivial connection on the trivial vector bundle $G \times \mathrm{End}(V) \to G$ (see Appendix A.3). Then $dg \in \Omega^1(G, \mathrm{End}(V))$ is an $\mathrm{End}(V)$-valued 1-form on G. This can be viewed as a map $dg : TG \to \mathrm{End}(V)$, and this is precisely the meaning of the symbol dg in (2). That said, tracing everything through, we see that since $g : G \to G$ is the identity map, this map $dg : TG \to \mathrm{End}(V)$ is the inclusion:

$$dg(v) = v$$

for $v \in T_gG \subseteq \mathrm{End}(V)$. Thus, (2) recovers (1), as it had better.

1.2 The left-invariant Maurer–Cartan form

In this section I will focus on the left-invariant version $\omega_L \in \Omega^1(G, \mathfrak{g})$. This has several key properties:

- This 1-form ω_L is vertical: Fix $\xi \in \mathfrak{g}$ and let be $\xi^{\#}$ is the vector field on G induced by right multiplication on G; that is

$$(\xi^{\#})_g := \frac{d}{d\tau}\Big|_{\tau=0} g\exp(t\xi).$$

  Then

$$\omega_L(\xi^{\#}) = \xi.$$

- This 1-form ω_L in equivariant: Fix $g \in G$ (recall that R_g is the diffeomorphism of G given by right multiplication). Then

$$R_g^*\omega_L = \mathrm{Ad}(g^{-1})\omega_L.$$

View G as a principal G-bundle over a point:

$$G \longrightarrow G/G = \{\mathrm{pt}\},$$

with G acting on itself by right multiplication. It follows from the properties above that ω_L defines a connection on this principal G-bundle. For a review of connections in the principal bundle setting, see Appendix A.4. Moreover, ω_L is the *only* connection on this bundle. This is because any two connections differ by a basic 1-form, and the space of basic 1-forms on $G \to \{\mathrm{pt}\}$ is isomorphic to $\Omega^1(\{\mathrm{pt}\}, \mathfrak{g}) = \{0\}$ (how many 1-forms are on a zero-dimensional manifold?).

In fact, every trivial connection can be viewed as a pullback of the left-invariant Maurer–Cartan connection: Suppose $P \to M$ is a trivializable principal G-bundle, and fix a trivialization $\tau : P \to M \times G$. Write $\pi_2 : M \times G \to G$ for the projection. Then the trivial connection on P (relative to this trivialization τ) is the pullback $(\pi_2 \circ \tau)^* \omega_L \in \Omega^1(P, \mathfrak{g})$ of ω_L. See Appendix A.5 for an overview of trivial connections in the principal bundle setting.

Example 1. *Consider the case where $G = \mathrm{SU}(2)$. Then we can write a generic element $g \in \mathrm{SU}(2)$ as*

$$g = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$$

where $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha|^2 + |\beta|^2 = 1$. Then

$$\omega_L = g^{-1} dg = \begin{pmatrix} \overline{\alpha} d\alpha + \overline{\beta} d\beta & -\overline{\alpha} d\overline{\beta} + \overline{\beta} d\overline{\alpha} \\ -\beta d\alpha + \alpha d\beta & \alpha d\overline{\alpha} + \beta d\overline{\beta} \end{pmatrix}$$

Here is the quaternion version of this same example: Let $S^3 \subseteq \mathbb{H}$ be the unit sphere in the space of quaternions; the multiplication on $\mathbb{H}$ restricts to S^3 to give it a group structure. It is convenient to write $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$. Then we can write a generic element of $g \in S^3$ as $\alpha + j\beta$ where $\alpha, \beta \in \mathbb{C}$ satisfy $|\alpha|^2 + |\beta|^2 = 1$ (note that the order matters in $j\beta$, since $j\beta = \overline{\beta} j$). Then we have

$$\omega_L = (\overline{\alpha} d\alpha + \overline{\beta} d\beta) + j(-\beta d\alpha + \alpha d\beta).$$

Now, the map

$$S^3 \longrightarrow \mathrm{SU}(2), \qquad \alpha + j\beta \longmapsto \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}$$

is a group isomorphism (this is where the order in $\alpha + j\beta$ matters) and this identifies the Maurer–Cartan forms on the two spaces. Of note is that this isomorhpism carries right multiplication on S^3 by $z \in S^1 \subseteq S^3$ to right multiplication $\mathrm{SU}(2)$ by the diagonal matrix

$$\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}. \tag{3}$$

Later on we will view S^3 and $\mathrm{SU}(2)$ as principal S^1-bundles via this right multiplication action; see also the next remark.

Remark 1. *For the moment, view S^3 as a subset of $\mathbb{C}^2$, as opposed to the technically-slightly-different space $\mathbb{H}$ like I did in the previous example. The* *scalar action of* $S^1 \subseteq \mathbb{C}$ *on* $\mathbb{C}^2$ *restricts to an action on* S^3*, and the* Hopf fibration *is by definition the quotient map*

$$\pi_H : S^3 \longrightarrow \mathbb{C}P^1 := S^1 \backslash S^3.$$

The point of this remark is to explain why the Hopf fibration corresponds exactly to the principal S^1*-bundle structure on the unit sphere in* $\mathbb{H}$ *defined in the previous example using* right multiplication *on* $\mathbb{H}$*. This is predicated on me having written the quaternions as* $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$ *with coordinates*

$$\mathbb{C} \oplus \mathbb{C} \longrightarrow \mathbb{H}, \qquad (\alpha, \beta) \longmapsto \alpha + j\beta.$$

Of course, this map identifies the unit spheres in both spaces, which is why I feel justified in using the same symbol S^3 *for both. However, it is a little funny with the* S^1*-action: This map takes scalar multiplication by* $z \in S^1$ *on* $\mathbb{C}^2$ *to* right multiplication *by* z *on* $\mathbb{H}$ *(there isn't a problem about left-versus-right actions here since multiplication on* $\mathbb{C}$ *is commutative). Thus, scalar multiplication on* $\mathbb{C}^2$ *corresponds to right multiplication on* $\mathbb{H} = \mathbb{C} + j\mathbb{C}$ *under these coordinates.*

If we were to use the coordinates $(\alpha, \beta) \mapsto \alpha + \beta j$*, then we would need to use left multiplication on* $\mathbb{H}$ *to get the Hopf fibration. The* SU(2)*-version is that the coordinates*

$$(\alpha, \beta) \longmapsto \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

intertwine scalar multiplication on $\mathbb{C}^2$ *with left multiplication by the diagonal matrix (3) on matrices. Moving forward, I will stick with the coordinates of Example 1 for my Hopf fibration.*

1.3 The right-invariant Maurer–Cartan form

Now let's consider $\omega_R \in \Omega^1(G, \mathfrak{g})$. When $\mathfrak{g}$ is not abelian, $\omega_R \neq \omega_L$. It then follows from the discussion of the previous section that ω_R is *not* a connection on the trivial bundle $G \to G/G = \{\text{pt}\}$ (it is this reason that I will focus on ω_L later, and not ω_R). The property that fails is equivariance: The 1-form $\omega_R : G \to \mathfrak{g}$ intertwines the right multiplication action of G on itself with the *trivial* action of G on $\mathfrak{g}$; the action on $\mathfrak{g}$ would have to be the adjoint action for ω_R to define a connection.

This asymmetry between ω_L and ω_R is an artifact of a convention: We had viewed G as a principal G-bundle over $\{\text{pt}\}$ by having G act on itself by *right* multiplication. If we had G act on itself by *left* multiplication, then we still get a principal G-bundle $G \to G \backslash G = \{\text{pt}\}$, but where the roles of ω_L and ω_R are reversed: ω_R is the only connection on this bundle and ω_L is not a connection.

Put more formally, write G_R for G viewed as a principal G-bundle with G acting by right multiplication, and write G_L for G viewed as a principal G-bundle with G acting by left multiplication (necessarily with an inverse thrown in this latter case to make sure we still have a right action). Then ω_L is the only connection on G_R and ω_R is the only connection on G_L. The map

$$G_R \longrightarrow G_L \qquad g \longmapsto g^{-1}$$

is an isomorphism of principal G-bundles that pulls ω_R back to ω_L.

2 A non-connection on TG

Assume $G \subseteq \mathrm{GL}(V)$ is a matrix group. Since $\mathrm{End}(V)$ is a vector space, its tangent bundle $T\mathrm{End}(V) \cong \mathrm{End}(V) \times \mathrm{End}(V)$ is canonically trivial. In this section, the symbol d will stand for the restriction to G of the trivial connection on this tangent bundle; thus, d can be viewed as a map of the form

$$d : \Omega^0(G, \mathrm{End}(V)) \longrightarrow \Omega^1(G, \mathrm{End}(V)).$$

Equivalently, the map d is the trivial connection on the trivial bundle $G \times \mathrm{End}(V) \to G$.

The point is that this operator d does not, at least in general, define a connection on TG. The issue is that there may be $\xi \in \Omega^0(G, TG) \subseteq \Omega^0(G, \mathrm{End}(V))$ with $d\xi \notin \Omega^1(G, TG)$; that is, d *does not* necessarily corestrict to define a map of the form

$$\Omega^0(G, TG) \longrightarrow \Omega^1(G, TG).$$

The next example gives probably the simplest counterexample illustrating this phenomenon.

Example 2. *Consider the case where $G = S^1 \subseteq \mathbb{C}$. Suppose $\xi \in \Omega^0(G, TG)$ is a vector field on G, so*

$$\xi(z) \in T_z G$$

for all $z \in G$; that is, $\xi(z)$ is orthogonal to z for all $z \in S^1$. Writing $z = e^{i\theta}$, we have $\xi(e^{i\theta}) = if(\theta)e^{i\theta}$ for some 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$. Then

$$d\xi = -f(\theta)e^{i\theta} + if'(\theta)e^{i\theta}.$$

However, this is not orthogonal to $e^{i\theta}$ unless f is identically zero. This shows $d\xi$ is not an element of $\Omega^1(G, TG)$ when $\xi \neq 0$.

3 Two connections on TG

The diffeomorphisms $L_g : G \to G$ and $R_g : G \to G$ each provide maps of the form

$$\begin{array}{rrcl} \Phi_L : & TG & \longrightarrow & G \times \mathfrak{g} \\ & T_gG \ni v & \longmapsto & (g, (L_{g^{-1}})_* v) \end{array} \qquad \begin{array}{rrcl} \Phi_R : & TG & \longrightarrow & G \times \mathfrak{g} \\ & T_gG \ni v & \longmapsto & (g, (R_{g^{-1}})_* v) \end{array}$$

That is,

$$\Phi_L = \pi_{TG} \times \omega_L, \qquad \Phi_R = \pi_{TG} \times \omega_R$$

where $\pi_{TG} : TG \to G$ is the projection and ω_L, ω_R are the Maurer–Cartan forms, but viewed as functions $TG \to \mathfrak{g}$. Viewing TG and $G \times \mathfrak{g}$ as vector bundles over G, the maps Φ_L and Φ_R each provide a trivialization of TG. In particular, each defines a connection on TG by pulling back the trivial connection. Since $TG \to G$ is a vector bundle, it is more natural to describe this in the language of vector bundles and linear connections; I will call the linear connections thus created ∇_L and ∇_R. I will describe both in detail, starting with "L"-version.

By definition, $\nabla_L : \Omega^0(G, TG) \to \Omega^1(G, TG)$ is the operator defined by

$$\nabla_L = \Phi_L^{-1} \circ d \circ \Phi_L,$$

where in this section d is the trivial connection on $G \times \mathfrak{g} \to G$. If we assume that $G \subseteq \mathrm{GL}(V)$ is a matrix group, then there is another way to write this: Letting $\xi \in \Omega^0(G, TG)$, we have

$$\begin{aligned} (\nabla_L(\xi))_g &= \Phi_L^{-1} \circ d(g, g^{-1}\xi) \\ &= \Phi_L^{-1}(g, g^{-1}d\xi - g^{-1}(dg)g^{-1}\xi) \\ &= d\xi - (dg)g^{-1}\xi. \end{aligned}$$

Thus

$$\nabla_L = d - (dg)g^{-1}. \tag{4}$$

Remark 2. *This same formula (4) also defines a linear connection ∇'_L on the trivial vector bundle $G \times \mathrm{End}(V) \to G$, and also a connection ∇''_L on $\mathrm{End}(V) \times \mathrm{End}(V) \to \mathrm{End}(V)$. Each of these bundles is naturally trivial, and the trivial connection is the operator d but viewed as a connection on $G \times \mathrm{End}(V)$ or on $\mathrm{End}(V) \times \mathrm{End}(V)$, depending on context. In particular, we see from (4) that the connection 1-form for ∇'_L and for ∇''_L is $-(dg)g^{-1}$.*

However, I want to emphasize that the connection 1-form of ∇_L (which is a connection on TG) is generally not *$-(dg)g^{-1}$. One way to see this is that the operator d in (4) is not a connection on TG, at least in general (this was the point of Section 2). Another way to see this is that, for a connection to define a connection 1-form, we need to pick a trivialization. An obvious choice of trivialization here is Φ_L, but the connection 1-form for ∇_L relative to this trivialization is $0 \in \Omega^1(M, \mathrm{End}(V))$ and this is typically not equal to $-(dg)g^{-1}$.*

Similarly, define ∇_R to be the connection on TG pulled back from the trivial connection via $\Phi_R : TG \to G \times \mathfrak{g}$. Then a computation similar to the one just given shows

$$\nabla_R = d - g^{-1}dg.$$

3.1 A digression on these trivializations

Consider the action of G on G by left multiplication. This induces an action of G on TG, and $\Phi_L : TG \to G \times \mathfrak{g}$ intertwines this with the diagonal action on $G \times \mathfrak{g}$ with *the trivial action* on $\mathfrak{g}$. On the other hand, $\Phi_R : TG \to G \times \mathfrak{g}$ intertwines the left G-action on TG with the diagonal action on $G \times \mathfrak{g}$ with *the adjoint action* on $\mathfrak{g}$. In just the same way, if we were to instead consider the action of G on G by *right* multiplication, then Φ_L would correspond to the adjoint action on $\mathfrak{g}$, and Φ_R would correspond to the trivial action on $\mathfrak{g}$. This is very much related to the fact that ω_L is a connection on G (viewed as a

G-bundle via right multiplication) while ω_R is not, with the situation reversing if we consider the left multiplication action of G on itself.

Interestingly, the connection ∇_L is equivariant under *both* the left- and the right-multiplication actions of G on itself:

$$(\nabla_L(h\xi))_{hg} = h((\nabla_L \xi)_g), \qquad (\nabla_L(\xi h))_{gh} = ((\nabla_L \xi)_g)h, \qquad \forall g, h \in G, \ \forall \xi \in \mathfrak{g}.$$

Similarly, ∇_R is also equivariant under both of these actions.

Example 3. *Consider the case where $G = S^3$ and let $\mathcal{V} \subseteq TG$ be the vertical bundle of the Hopf fibration $\pi_H : S^3 \to \mathbb{C}P^1$; see Remark 1 and Example 1. As in that example, I will view $S^3 \subseteq \mathbb{H}$ as a principal S^1-bundle where S^1 acts on S^3 by right multiplication. The Lie algebra $\mathfrak{g} = \mathrm{span}(i, j, k)$ consists of the purely imaginary quaternions. The fundamental vector field of the S^1-action is given by $\xi^\#(g) = gi$; this spans $\mathcal{V}$. The orthogonal complement $\mathcal{V}^\perp \subseteq TS^3$ can be identified with the pullback bundle $\pi_H^* TS^2 \to S^3$. The basic forms are those on S^3 that take values in $\mathcal{V}^\perp$ and are equivariant relative to the given S^1-action on S^3 and its induced action on TG (which preserves $\mathcal{V}^\perp$). Note that the adjoint action of S^1 on $\mathfrak{g}$ fixes i and is a rotation in the jk-plane; the map $S^1 \to \mathrm{SO}(\mathrm{span}(i, j))$ is two-to-one.*

If we use Φ_L to trivialize TG, then the fundamental vector field $\xi(g) = gi$ is taken to the constant vector field $G \to \mathfrak{g}$ sending $g \mapsto i$. Since Φ_L intertwines the given action of TG with the action on $G \times \mathfrak{g}$ acting trivially on $\mathfrak{g}$, the basic 0-forms can be identified with the S^1-invariant functions $\psi : G \mapsto \mathfrak{g}$ with $\psi(g) \perp i$ for all $g \in G$.

If we use Φ_R to trivialize TG, then gi is taken to the nonconstant vector field $G \to \mathfrak{g}$ sending $g \mapsto gig^{-1}$. Since Φ_R intertwines the given action of TG with the action on $G \times \mathfrak{g}$ acting by the adjoint on $\mathfrak{g}$, the basic 0-forms can be identified with the functions $\psi : G \mapsto \mathfrak{g}$ with $\psi(g) \perp gig^{-1}$ for all g that are equivariant relative to the adjoint action of S^1 on $\mathfrak{g}$.

4 Homogeneous spaces

Suppose $\phi : K \hookrightarrow G$ is an embedding of a Lie group K with Lie algebra $\mathfrak{k}$. Assume that $\mathfrak{g}$ has an Ad-invariant inner product. The construction below depends on this inner product modulo conformal scaling; thus, if $\mathfrak{g}$ is simple then the construction is independent of the inner product. I want to think of G as a principal K-bundle over G/K (with K acting by right multiplication).

Write $\omega \in \Omega^1(G, \mathfrak{g})$ for the left-invariant Maurer–Cartan form, and let $\phi_* : \mathfrak{k} \to \mathfrak{g}$ be the pushforward of ϕ at the identity. Note that ϕ_* is a Lie algebra embedding. In particular it corestricts to an isomorphism $\phi_*| : \mathfrak{k} \to \mathrm{im}(\phi_*)$, and I will write $\phi_*^{-1} : \mathrm{im}(\phi_*) \to \mathfrak{k}$ for the inverse of this corestriction. Define a 1-form $A_K^G := A_K \in \Omega^1(G, \mathfrak{k})$ by the formula

$$A_K := (\phi_*)^{-1} \mathrm{proj}_{\mathrm{im}(\phi_*)} \omega$$

where $\mathrm{proj}_{\mathrm{im}(\phi_*)} : \mathfrak{g} \to \mathrm{im}(\phi_*)$ is the orthogonal projection. I claim that A_K is a connection on the principal K-bundle $G \to G/K$. Let's check the axioms:

- Vertical: Let $\xi \in \mathfrak{k}$. The induced vector field on G is given by

$$\xi^{\#}(g) = \frac{d}{d\tau}\Big|_{\tau=0} g\phi(\exp(\tau\xi)).$$

We have $\phi(\exp(\tau\xi)) = \exp(\tau\phi_*\xi)$. Since $\phi_*\xi \in \mathrm{im}(\phi_*) \subseteq \mathfrak{g}$ we have

$$\begin{aligned} A_K(\xi^{\#}) &= (\phi_*)^{-1}\mathrm{proj}_{\mathrm{im}(\phi_*)}\omega(\xi^{\#}) \\ &= (\phi_*)^{-1}\mathrm{proj}_{\mathrm{im}(\phi_*)}\phi_*\xi \\ &= (\phi_*)^{-1}\phi_*\xi \\ &= \xi. \end{aligned}$$

- Equivariant: For all $k \in K$,

$$R_k(g) = g\phi(k),$$

where $R_k : G \to G$ is right multiplication by k. Then $\phi(k) \in G$, so the equivariance of ω gives

$$\begin{aligned} R_k^* A_K &= (\phi_*)^{-1}\mathrm{proj}_{\mathrm{im}(\phi_*)} R_k^*\omega \\ &= (\phi_*)^{-1}\mathrm{proj}_{\mathrm{im}(\phi_*)}\mathrm{Ad}(\phi(k)^{-1})\omega \\ &= (\phi_*)^{-1}\mathrm{Ad}(\phi(k)^{-1})\mathrm{proj}_{\mathrm{im}(\phi_*)}\omega \\ &= \mathrm{Ad}(k^{-1})(\phi_*)^{-1}\mathrm{proj}_{\mathrm{im}(\phi_*)}\omega \\ &= \mathrm{Ad}(k^{-1})A_K. \end{aligned}$$

The fact that $\mathrm{Ad}(\phi(k)^{-1})$ commutes with the projection follows because the inner product is Ad-invariant.

Of course, when $K = G$ and ϕ is the identity, then $A_K = \omega$ is the Maurer–Cartan form.

Example 4. *Consider the case where $G = \mathrm{SU}(2)$ and $K = S^1$, with $\phi : S^1 \to \mathrm{SU}(2)$ defined by*

$$\phi(k) := \begin{pmatrix} k & 0 \\ 0 & \overline{k} \end{pmatrix}.$$

View S^1 as acting on $\mathrm{SU}(2)$ by right multiplication, as in Example 1. As we saw in that example and Remark 1, this is bundle isomorphic to the Hopf fibration, but it is instructive to see how the details of this section work out in this $\mathrm{SU}(2)$-setting since ϕ is not an inclusion.

By Example 1 the Maurer–Cartan form can be written as

$$\omega = g^{-1}dg = \begin{pmatrix} \overline{\alpha}d\alpha + \overline{\beta}d\beta & -\overline{\alpha}d\overline{\beta} + \overline{\beta}d\overline{\alpha} \\ -\beta d\alpha + \alpha d\beta & \alpha d\overline{\alpha} + \beta d\overline{\beta} \end{pmatrix}.$$

The projection to $\mathrm{im}(\phi_*)$ projects to the diagonal, so we have

$$\mathrm{proj}_{\mathrm{im}(\phi_*)}\omega = \begin{pmatrix} \overline{\alpha}d\alpha + \overline{\beta}d\beta & 0 \\ 0 & \alpha d\overline{\alpha} + \beta d\overline{\beta} \end{pmatrix}.$$

Then $(\phi_*)^{-1}$ *strips off the first component:*

$$A_K = \overline{\alpha} d\alpha + \overline{\beta} d\beta.$$

Now let's compute the curvature F_{A_K} *of this. Using the notation of Appendix A.7, we have*

$$\begin{aligned} \tilde{F}_{A_K} &= dA_K \\ &= -d\alpha \wedge d\overline{\alpha} - d\beta \wedge d\overline{\beta}. \end{aligned}$$

Being the curvature of a connection, this is a basic 2-form on the base $\mathrm{SU}(2)/S^1 \cong \mathbb{C}P^1$*. Thus, there is a unique 2-form* F_{A_K} *on* $\mathbb{C}P^1$ *with*

$$\pi^* F_{A_K} = \tilde{F}_{A_K}.$$

where $\pi : \mathrm{SU}(2) \to \mathbb{C}P^1$ *is the projection. Since* K *is abelian, the adjoint bundle is trivial, so this 2-form is taking values in the trivial vector bundle with fiber* $i\mathbb{R} = \mathrm{Lie}(K)$*; that is,* F_{A_K} *is a 2-form on* $\mathbb{C}P^1$ *with values in* $i\mathbb{R}$*. There are many ways to compute this, but here is a fun one using homogeneous coordinates. Specifically, consider the coordinate chart* $\psi : \mathbb{C} \to \mathbb{C}P^1$ *given by sending* $z \in \mathbb{C}$ *to* $[z, 1]$*. I will show that* F_{A_K} *is the pullback via* ψ^{-1} *of the 2-form*

$$-\frac{1}{(1+|z|^2)^2} dz \wedge d\overline{z} = \frac{2i}{(1+x^2+y^2)^2} dx \wedge dy$$

on $\mathbb{C}$*, where* $z = x + iy$ *are the obvious coordinates on* $\mathbb{C}$*. To see this, the uniqueness of* F_{A_K} *implies we just need to show*

$$(\psi^{-1} \circ \pi)^* \frac{1}{(1+|z|^2)^2} dz \wedge d\overline{z} = \tilde{F}_{A_K}.$$

To see this, note that

$$z = (\psi^{-1} \circ \pi) \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} = \alpha / \beta.$$

(We are working in the chart where $\beta \neq 0$*.) Then* $dz = \beta^{-2}(\beta d\alpha - \alpha d\beta)$*. Expanding and then using* $|\alpha|^2 + |\beta|^2 = 1$*, we have*

$$\begin{aligned} (\psi^{-1} \circ \pi)^* dz \wedge d\overline{z} &= |\beta|^{-4} \Big(|\beta|^2 d\alpha \wedge d\overline{\alpha} + |\alpha|^2 d\beta \wedge d\overline{\beta} \\ &\qquad -2\mathrm{Re}(\overline{\alpha}\beta d\alpha \wedge d\overline{\beta}) \Big) \\ &= |\beta|^{-4} \Big(d\alpha \wedge d\overline{\alpha} + d\beta \wedge d\overline{\beta} \\ &\qquad -|\alpha|^2 d\alpha \wedge d\overline{\alpha} - |\beta|^2 d\beta \wedge d\overline{\beta} \\ &\qquad -2\mathrm{Re}(\overline{\alpha}\beta d\alpha \wedge d\overline{\beta}) \Big) \\ &= |\beta|^{-4} \Big(d\alpha \wedge d\overline{\alpha} + d\beta \wedge d\overline{\beta} \\ &\qquad -(\overline{\alpha} d\alpha + \overline{\beta} d\beta) \wedge (\alpha d\overline{\alpha} + \beta d\overline{\beta}) \Big). \end{aligned}$$

Differentiating $|\alpha|^2 + |\beta|^2 = 1$, we find

$$\overline{\alpha} d\alpha + \overline{\beta} d\beta = -(\alpha d\overline{\alpha} + \beta d\overline{\beta})$$

and so this last term vanishes. This gives

$$(\psi^{-1} \circ \pi)^* dz \wedge d\overline{z} = |\beta|^{-4} \Big(d\alpha \wedge d\overline{\alpha} + d\beta \wedge d\overline{\beta} \Big),$$

from which we conclude

$$\begin{aligned} (\psi^{-1} \circ \pi)^* \frac{1}{(1+|z|^2)^2} dz \wedge d\overline{z} &= d\alpha \wedge d\overline{\alpha} + d\beta \wedge d\overline{\beta} \\ &= -\tilde{F}_A \end{aligned}$$

as claimed.

From here, we can compute the first Chern number of the S^1-bundle $\mathrm{SU}(2) \to \mathbb{C}P^1$ *via the Chern–Weil formula:*

$$\begin{aligned} c_1(S^3) \frown [\mathbb{C}P^1] &= \frac{i}{2\pi} \int_{\mathbb{C}P^1} F_{A_K} \\ &= \frac{i}{2\pi} \int_{\mathbb{C}} (\psi^{-1})^* F_{A_K} \\ &= -\frac{1}{\pi} \int_{\mathbb{C}} \frac{1}{(1+x^2+y^2)^2} dx \wedge dy \\ &= -1. \end{aligned}$$

This is as expected (the Hopf fibration is the frame bundle for the tautological line bundle on $\mathbb{C}P^1$, which is $\mathcal{O}(-1)$).

5 Action on another manifold

Suppose $K \subseteq G$ is a subgroup and M is a manifold equipped with a right action of K. Then K acts freely on $G \times M$ (because it acts freely on the first factor) and we can consider it as a principal K-bundle over $(G \times M)/K$. Write

$$A_{K;M} := \pi_G^* A_K$$

where π_G is the projections from $G \times M$ to G. That this is a connection can be checked directly, but here is an alternative approach, rooted in the observation that this is literally a pullback connection: View $G \to G/K$ as a principal K-bundle. The map $\pi_G : G \times M \to G$ is K-equivariant and so descends to a map $\pi : (G \times M)/K \to G/K$. Then the pullback bundle $\pi^* G \to (G \times M)/K$ is naturally isomorphic, as a principal K-bundle, to $G \times M \to (G \times M)/K$ and this isomorphism identifies the pullback connection of $\pi^* A_K$ with the connection denoted $\pi_G^* A_K$ above. An upshot of this approach is that it shows the curvature of $A_{K;M}$ is the π-pullback of the curvature of A_K:

$$\pi^* F_{A_K} = F_{A_{K;M}}.$$

6 Products of Lie groups

Now suppose K, G_0, G_1 are Lie groups and

$$\phi_j : K \longrightarrow G_j$$

is an embedding for $j = 0, 1$. Then we can consider $G_0 \times G_1$ as a principal K-bundle with the diagonal action. Fix $1 \leq p, q \leq \infty$ with $p^{-1} + q^{-1} = 1$. Write

$$A_{p,q} := p^{-1}\pi_0^* A_K^{G_0} + q^{-1}\pi_1^* A_K^{G_1}$$

where π_j is the projections from $G_0 \times G_1$ to G_j. I claim that this is a connection.

- Vertical: Fix $\xi \in K$ and write $\xi^\#$ for the induced vector field on $G_0 \times G_1$. Temporarily write $\xi_j^\#$ for the induced vector field on G_j. Note that π_j is K-equivariant, so $(\pi_j)_* \xi^\# = \xi_j^\#$. This implies

$$\begin{aligned} A_{p,q}(\xi^\#) &= p^{-1} A_K^{G_0}(\xi_0^\#) + q^{-1} A_K^{G_1}(\xi_1^\#) \\ &= p^{-1}\xi + q^{-1}\xi \\ &= \xi. \end{aligned}$$

- Equivariant: Fix $k \in K$. Letting $(R_k)_j : G_j \to G_j$ be right multiplication by $\phi_j(k)$ on G_j, and $R_k : G \to G$ right multiplication by $(\phi_0(k), \phi_1(k))$, the equivariance of the π_j implies $\pi_j \circ R_k = (R_k)_j \circ \pi_j$ and this gives

$$\begin{aligned} R_k^* A_{p,q} &= p^{-1}\pi_0^*(R_k)_0^* A_K^{G_0} + q^{-1}\pi_1^*(R_k)_1^* A_K^{G_1} \\ &= p^{-1}\pi_0^* \mathrm{Ad}(k^{-1}) A_K^{G_0} + q^{-1}\pi_1^* \mathrm{Ad}(k^{-1}) A_K^{G_1} \\ &= \mathrm{Ad}(k^{-1}) A_{p,q}. \end{aligned}$$

Example 5. *Consider the case where $p = 1$ and $q = \infty$. Then*

$$A_{1,\infty} = \pi_0^* A_K^{G_0}.$$

This recovers the construction of Section 5.

Example 6. *Consider the case where $G := K = G_0 = G_1$ with $\phi_0 = \phi_1 = \mathrm{Id}_G$ the identity map. Then*

$$A_{p,q} = p^{-1}\pi_0^* A^G + q^{-1}\pi_1^* A^G,$$

so

$$A_{p,q}(g_0\xi_0, g_1\xi_1) = p^{-1}\xi_0 + q^{-1}\xi_1.$$

Let's compute the curvature of this thing. We have

$$\begin{aligned} dA_{p,q} &= p^{-1}\pi_0^* dA^{G_0} + q^{-1}\pi_1^* dA^{G_1} \\ &= -\tfrac{1}{2p}\pi_0^*[A^{G_0} \wedge A^{G_0}] - \tfrac{1}{2q}\pi_1^*[A^{G_1} \wedge A^{G_1}] \end{aligned}$$

We also have

$$\begin{aligned} \tfrac{1}{2}\left[A_{p,q}\wedge A_{p,q}\right] &= \tfrac{1}{2p^2}[\pi_0^*A^{G_0}\wedge\pi_0^*A^{G_0}] + \tfrac{1}{2q^2}[\pi_q^*A^{G_q}\wedge\pi_q^*A^{G_q}] \\ &\quad + \tfrac{1}{pq}[\pi_0^*A^{G_0}\wedge\pi_1^*A^{G_1}] \end{aligned}$$

Thus,

$$\begin{aligned} \tilde{F}_{A_{p,q}} &= \tfrac{1-p}{p^2}\pi_0^*[A^{G_0}\wedge A^{G_0}] + \tfrac{1-q}{q^2}\pi_1^*[A^{G_1}\wedge A^{G_1}] \\ &= +\tfrac{1}{pq}[\pi_0^*A^{G_0}\wedge\pi_1^*A^{G_1}]. \end{aligned}$$

Somewhat more explicitly, this is the 2-form that sends the pair

$$(g_0\xi_0, g_1\xi_1), (g_0\xi_0', g_1\xi_1') \in T_{(g_0,g_1)}G_0\times G_1$$

of tangent vectors to

$$\frac{1-p}{p^2}[\xi_0,\xi_0'] + \frac{1-q}{q^2}[\xi_1,\xi_1'] + \frac{1}{pq}([\xi_0,\xi_1'] + [\xi_1,\xi_0']).$$

To compute $F_{A_{p,q}}$, note that the G-bundle $G^2 = G_0\times G_1$ is trivializable with trivialization $G_0\times G_1 \to G\times G$ sending $(g_0,g_1)\mapsto(g_0g_1^{-1},g_1)$ (K acting diagonally by right multiplication on $G_0\times G_1$ and on $G\times G$ by right multiplication on the second factor). At the Lie algebra level, this sends $(v_0,v_1)\in T_{g_0}G_0\times T_{g_1}G_1$ to $(v_0g_1^{-1} - g_0g_1^{-1}v_1g_1^{-1}, v_1)\in T_{g_0g_1^{-1}}G_0\times T_{g_1}G_1$. The inverse map is

$$\begin{aligned} T_{g_0g_1^{-1}}G_0\times T_{g_1}G_1 &\longrightarrow T_{g_0}G_0\times T_{g_1}G_1 \\ (w_0,w_1) &\longmapsto (w_0g_1 + g_0g_1^{-1}w_1, w_1). \end{aligned}$$

Pulling the 2-form $\tilde{F}_{A_{p,q}}$ back under this map, we obtain the 2-form on $G\times G$ that sends the pair $(w_0,w_1),(w_0',w_1')\in T_{g_0g_1^{-1}}G\times T_{g_1}G$ to

$$\begin{aligned} &\tfrac{1-p}{p^2}\Big[g_0^{-1}w_0g_1 + g_1^{-1}w_1, g_0^{-1}w_0'g_1 + g_1^{-1}w_1'\Big] \\ &\quad + \tfrac{1-q}{q^2}[g_1^{-1}w_1, g_1^{-1}w_1'] \\ &\quad + \tfrac{1}{pq}\Big(\Big[g_0^{-1}w_0g_1 + g_1^{-1}w_1, g_1^{-1}w_1'\Big] + \Big[g_1^{-1}w_1, g_0^{-1}w_0'g_1 + g_1^{-1}w_1'\Big]\Big) \\ &= \tfrac{1-p}{p^2}[g_0^{-1}w_0g_1, g_0^{-1}w_0'g_1] \\ &\quad + \left(\tfrac{1-p}{p^2} + \tfrac{1-q}{q^2} + \tfrac{2}{qp}\right)[g_1^{-1}w_1, g_1^{-1}w_1'] \\ &\quad + \left(\tfrac{1-p}{p^2} + \tfrac{1}{pq}\right)[g_1^{-1}w_1, g_0^{-1}w_0'g_1] \\ &\quad + \left(\tfrac{1-p}{p^2} + \tfrac{1}{pq}\right)[g_0^{-1}w_0g_1, g_1^{-1}w_1'] \\ &= \tfrac{1-p}{p^2}[g_0^{-1}w_0g_1, g_0^{-1}w_0'g_1] \end{aligned}$$

where the last equality follows because $1/p + 1/q = 1$ (all three last terms go away). This is a 2-form supported on the first component of $G\times G$, which is *what we expect: the curvature is supposed to be basic. That is, the curvature (relative to this trivialization of the adjoint bundle) is*

$$F_{A_{p,q}}(g_0\zeta_0 g_1^{-1}, g_0\zeta_0' g_1^{-1}) = \frac{1-p}{p^2}[\zeta_0, \zeta_0'] = -\frac{1}{pq}[\zeta_0, \zeta_0'],$$

where we are using g_0 on the left and g_1^{-1} on the right to trivialize $T_{g_0 g_1^{-1}}G$. (Take $g_0 = g$ and $g_1 = e$ to get a trivialization relative to left multiplication.) This shows that the curvature depends on p: when $p = 1$ or $p = \infty$ it is flat (as it should be because it is just the pullback of the Maurer–Cartan form in that case). It is not flat when, say, $p = 2$.

A An overview of connections

Here I give [insert section title here], highlighting definitions but skipping essentially all proofs of my assertions. There are plenty of more detailed references for this material. If you are new to it, I strongly suggest Spivak, Vol. II for the basics.

A.1 Bundle-valued forms

Given a vector bundle $E \to M$ over a manifold M, I will write $\Omega^k(M, E)$ for the space of E-valued k-forms on M. If $E = M \times V$ is a trivial vector bundle, with V a vector space, then I may write $\Omega^k(M, V) := \Omega^k(M, M \times V)$. For example, the elements of $\Omega^0(M, V)$ can be viewed as smooth functions $M \to V$, and the elements of $\Omega^1(M, V)$ can be viewed as smooth functions $TM \to V$.

A.2 Connections on vector bundles

A *(linear) connection* on $E \to M$ is an $\mathbb{R}$-linear map

$$\nabla : \Omega^0(M, E) \longrightarrow \Omega^1(M, E)$$

satisfying the Leibniz rule:

$$\nabla(f\phi) = df \otimes \xi + f\nabla\xi, \qquad \forall f \in \Omega^0(M, \mathbb{R}), \qquad \forall \phi \in \Omega^0(M, E).$$

I will also refer to ∇ as a *covariant derivative.*

Suppose $E' \to M'$ is another vector bundle and $F : E' \to E$ is a bundle map covering a smooth map $f : M' \to M$. Assume further that F restricts to each fiber to be an invertible linear transformation. If ∇ is a linear connection on E, then we can create a linear connection $F^*\nabla$ on E' called the *pullback of* ∇. This is defined by

$$F^*\nabla := F \circ \nabla \circ F^{-1}$$

where the inverse is the fiberwise inverse of F.

Suppose ∇ is a connection on $E \to M$ and $\eta \in \Omega^1(M, \mathrm{End}(E))$. Then the sum $\nabla + \eta$ is again a connection on E. Conversely, if ∇ and ∇' are two connections on E, then the difference $\nabla - \nabla'$ satisfies $(\nabla - \nabla')(f\phi) = f(\nabla - \nabla')\phi$ for all $f \in \Omega^0(M, \mathbb{R})$ and $\phi \in \Omega^0(M, E)$. This implies that $\nabla - \nabla' \in \Omega^1(M, \mathrm{End}(E))$ is a 1-form on M with values in the bundle $\mathrm{End}(E)$. It follows that the space of linear connections on E is an affine space modeled on $\Omega^1(M, \mathrm{End}(E))$.

A.3 The trivial connection in the vector bundle setting

Suppose $E = M \times V$ is a trivial vector bundle. There is a canonically-defined connection on $M \times V$ called the *trivial connection*, which I will denote by

$$d : \Omega^0(M, V) \longrightarrow \Omega^1(M, V).$$

There are many ways to define this thing, and I will start with a low-brow approach. Pick a basis $v_1, \ldots, v_n$ for V. Then each $\phi \in \Omega^0(M, V)$ can be written as $\phi = \sum_i \phi_i v_i$ for some real valued functions $\phi_1, \ldots, \phi_n : M \to \mathbb{R}$. Then the *trivial connection* is defined by

$$d\phi := \sum_i (d\phi_i) \otimes w_i,$$

where the d appearing on the right is the de Rham operator. One can check that this is independent of the basis. Another fun way to write this is to use the basis to identify $V \cong \mathbb{R}^n$. This identifies

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}$$

with a vector, and then

$$d\phi = \begin{pmatrix} d\phi_1 \\ d\phi_2 \\ \vdots \\ d\phi_n \end{pmatrix}$$

is a vector of 1-forms.

Remark 3. *There is a slight variation people often adopt in the case where $V = \mathrm{End}(W)$ is the endomorphism ring of a vector space W. Pick a basis for W. This allows us to identify $\mathrm{End}(W)$ with the set of $n \times n$-matrices. In this way we can think of any $\phi \in \Omega^0(G, \mathrm{End}(W))$ as a matrix-valued function on G:*

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{pmatrix} : M \longrightarrow \{n \times n\text{-matrices}\},$$

where each $\phi_{ij} : M \to \mathbb{R}$ is a smooth function. Then $d\phi$ is the matrix

$$d\phi = \begin{pmatrix} d\phi_{11} & d\phi_{12} & \dots & d\phi_{1n} \\ d\phi_{21} & d\phi_{22} & \dots & d\phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d\phi_{n1} & d\phi_{n2} & \dots & d\phi_{nn} \end{pmatrix}$$

of 1-forms, where the d appearing on the right is the de Rham operator on real-valued functions.

Continue to assume that $E = M \times V$ is trivial, and let ∇ be any connection on E. By the discussion of the previous section, there is a unique 1-form $\eta \in \Omega^1(M, \mathrm{End}(V))$ so that

$$\nabla = d + \eta.$$

This η is often referred to as the *connection 1-form*, though it is important to keep in mind that it is dependent on the specific choice of trivialization of E. See Remark 4 for a further discussion.

A.4 Connections on principal bundles

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $\pi : P \to M$ is a principal G-bundle; when P is compact, this is equivalent to assuming that G acts freely on P on the right. The *vertical bundle* is the subbundle $\mathcal{V} := \ker(\pi_*) \subseteq TP$ given by the kernel of the pushforward π_*. This is canonically trivializable, with trivialization given by

$$\begin{array}{rcl} P \times \mathfrak{g} & \longrightarrow & \mathcal{V} \\ (p, \xi) & \longmapsto & \xi^{\#}(p) := \frac{d}{dt}\Big|_{t=0} p \exp(t\xi). \end{array}$$

A *(Ehresmann) connection* on P is a choice of K-invariant splitting of the sequence

$$0 \longrightarrow \mathcal{V} \hookrightarrow TP \xrightarrow{\pi_*} TM \longrightarrow 0.$$

There are various equivalent way to encode this information. One is as a G-equivariant bundle map $TP \to \mathcal{V}$ that restricts to the identity on $\mathcal{V}$. Since the vertical bundle is canonically-trivializable, such a bundle map is equivalent to a map $A : TP \to \mathfrak{g}$ that is equivariant

$$R_g^* A = \mathrm{Ad}(g^{-1}) A, \qquad \forall g \in G$$

and vertical

$$\iota_{\xi^{\#}} A = \xi, \qquad \forall \xi \in \mathfrak{g};$$

here $R_g : P \to P$ is the diffeomorphism given by right multiplication by g. I generally find this map A the most useful, so when I use the word "connection" in a principal bundle setting I am usually referring to it. Note that a connection A is a 1-form on P with values in $\mathfrak{g}$:

$$A \in \Omega^1(P, \mathfrak{g}).$$

Another way to encode the data of a connection is via a horizontal bundle. Specifically, an Ehresmann connection is equivalent to prescribing a subbundle $\mathcal{H} \subseteq TP$ that is G-equivariant

$$(R_g)_* \mathcal{H} = \mathcal{H}, \qquad g \in G$$

and that is horizontal in the sense that

$$TP = \mathcal{V} \oplus \mathcal{H}.$$

This $\mathcal{H}$ is the *horizontal bundle* associated to the Ehresmann connection, though sometimes people use the term "connection" or "Ehresmann connection" to refer to $\mathcal{H}$. Its relationship with A is given by

$$\mathcal{H} = \ker(A).$$

Now suppose $P' \to M'$ is another principal G-bundle and $F : P' \to P$ is an equivariant bundle map covering some smooth map $f : M' \to M$. If $A \in \Omega^1(P, \mathfrak{g})$ is a connection on $\pi : P \to M$, then so too is its pullback $F^*A \in \Omega^1(P', \mathfrak{g})$ (pullback defined in the usual way one pulls back a differential form). A particularly useful application of this is when we start with a smooth function $f : M' \to M$ (but not a bundle P'). Then we can create the *pullback bundle*

$$f^*P := \{(x, p) \in M' \times P \mid F(x) = \pi(p)\}.$$

This is naturally a principal G-bundle over M', and the projection to the second component gives a bundle map $F : f^*P \to P$ covering f. People often set

$$f^*A := F^*A \in \Omega^1(f^*P, \mathfrak{g})$$

and call A the *pullback* connection on f^*P. As an example, if $P = M \times G$ is trivial, then $f^*P = M' \times G$ is also trivial and $F(x, g) = (f(x), g)$.

A k-form $\phi \in \Omega^k(P, \mathfrak{g})$ is called *basic* if it is equivariant

$$R_g^*\phi = \mathrm{Ad}(g^{-1})\phi, \qquad \forall g \in G$$

and *horizontal*

$$\iota_{\xi^\#}\phi = 0, \qquad \forall \xi \in \mathfrak{g}.$$

Write

$$\Omega^\ell(P, \mathfrak{g})_{basic}$$

for the set of all *basic* ℓ-forms on P. This is a vector space. Write

$$P(\mathfrak{g}) := P \times_{\mathrm{Ad}} \mathfrak{g} = \frac{P \times \mathfrak{g}}{(p, \xi) \sim (pg, \mathrm{Ad}(g^{-1})\xi)}$$

for the bundle over M associated to the adjoint representation of G on $\mathfrak{g}$. It is a standard result (and a good exercise in notation-juggling) that there is a natural bundle isomorphism

$$\pi^*P(\mathfrak{g}) \cong P \times \mathfrak{g}.$$

At the level of forms, the pullback gives a vector space isomorphism

$$\pi^* : \Omega^k(M, P(\mathfrak{g})) \xrightarrow{\cong} \Omega^k(P, \mathfrak{g})_{basic}.$$

Suppose A and A' are two connections on P. Then the difference $A - A'$ is a basic 1-form. As such, it follows that $A - A' = \pi^*\eta$ for a unique $\eta \in \Omega^1(M, P(\mathfrak{g}))$. Conversely, if A is a connection on P and $\eta \in \Omega^1(M, P(\mathfrak{g}))$, then $A + \pi^*\eta$ is another connection on P. In this way, the space of connections on P is an affine space modeled on the vector space $\Omega^1(M, P(\mathfrak{g}))$.

A.5 The trivial connection in the principal bundle setting

Suppose $P = M \times G$ is the trivial principal G-bundle. Then the tangent bundle

$$TP = \pi_1^* TM \oplus \pi_2^* TG \tag{5}$$

naturally splits, where $\pi_1 : M \times G \to M$ and $\pi_2 : M \times G \to G$ are the projections. The second factor $\pi_2^* TG = \mathcal{V}$ is the vertical bundle. In this setting, the *trivial Ehresmann connection* is defined to be the one with horizontal bundle $\mathcal{H} = \pi_1^* TM$, so the projection $TP \to \mathcal{V} = \pi_2^* TG$ is exactly the one provided by the direct sum decomposition (5). Write $A_{\text{triv}} \in \Omega^1(P, \mathfrak{g})$ for the connection of the trivial Ehresmann connection; I will call this the *trivial connection* on $M \times G$.

Suppose P is trivial and A is any connection on P. As we saw above, the difference $A - A_{\text{triv}}$ is a basic 1-form and so there is a unique $\eta \in \Omega^1(M, \mathfrak{g})$ so that difference $A = A_{\text{triv}} + \pi^*\eta$. Since A_{triv} is canonically determined by the trivialization of P, people often identify A and η, and refer to this η as the *connection 1-form.*

Remark 4. *(a) A connection 1-form is a 1-form on M, while the connection is a 1-form on P. We can only get a connection 1-form when there is a trivialization of the bundle floating around.*

(b) In this way, one also can create from a connection a local *connection 1-form on M by choosing a local trivialization of P.*

A.6 Getting linear connections from connections on principal bundles

Now I will briefly describe how connections in the principal bundle setting produce linear connections. For this, consider the trivial linear connection $d : \Omega^0(P, \mathfrak{g}) \longrightarrow \Omega^1(P, \mathfrak{g})$ on the trivial vector bundle $P \times \mathfrak{g} \to P$. Given $\xi \in \Omega^k(P, \mathfrak{g})$ and $\zeta \in \Omega^\ell(P, \mathfrak{g})$, write $[\xi \wedge \zeta] \in \Omega^{k+\ell}(P, \mathfrak{g})$ for the form obtained by wedging on the form parts of ξ, ζ and taking the Lie bracket of the Lie algebra parts. This wedge-bracket is a bilinear map of the form

$$\Omega^k(P, \mathfrak{g}) \otimes \Omega^\ell(P, \mathfrak{g}) \longrightarrow \Omega^{k+\ell}(P, \mathfrak{g}).$$

It preserves the basic forms, and so determines a product structure

$$\Omega^k(M, P(\mathfrak{g})) \otimes \Omega^\ell(M, P(\mathfrak{g})) \longrightarrow \Omega^{k+\ell}(M, P(\mathfrak{g}))$$

on the space of $P(\mathfrak{g})$-valued forms on M.

If $A \in \Omega^1(P, \mathfrak{g})$ is a connection on P, then the formula $d + A$ defines a map $\Omega^0(P, \mathfrak{g}) \to \Omega^1(P, \mathfrak{g})$, with A acting by $[A \wedge \cdot]$. One can check $d + A$ preserves the basic forms and so determines an operator

$$d_A : \Omega^0(M, P(\mathfrak{g})) \longrightarrow \Omega^1(M, P(\mathfrak{g}))$$

via the diagram

$$\begin{array}{ccc} \Omega^0(P, \mathfrak{g})_{basic} & \xrightarrow{d+A} & \Omega^1(P, \mathfrak{g})_{basic} \\ {\scriptstyle \pi^*}\uparrow & & {\scriptstyle \pi^*}\uparrow \\ \Omega^0(M, P(\mathfrak{g})) & \xrightarrow{d_A} & \Omega^1(M, P(\mathfrak{g})) \end{array}$$

The operator d_A satisfies the Leibniz rule and so is a linear connection on the vector bundle $P(\mathfrak{g})$. At this point the word "connection" is obviously way overused, which is why I often refer to d_A as the *covariant derivative* associated to A. You can also check (depending on how bored you are) that this process intertwines the notions of pullback in the two settings. That is, if $F : P' \to P$ is a bundle isomorphism, then $d_{F^*A} = F^{-1} \circ d_A \circ F$.

The trivial connection in the principal bundle setting recovers the trivial connection in the vector bundle setting as follows: If $P = M \times G$ is trivial, then the adjoint bundle $P(\mathfrak{g}) = M \times \mathfrak{g}$ is canonically trivial, and the covariant derivative

$$d_{A_{\mathrm{triv}}} = d : \Omega^0(M, \mathfrak{g}) \longrightarrow \Omega^1(M, \mathfrak{g})$$

associated to A_{triv} is the trivial connection d on the trivial vector bundle $M \times \mathfrak{g}$.

There is also a way of going from a linear connection $\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$ to a connection on a principal bundle. The principal bundle will be an appropriate frame bundle of E. For example, suppose E is a real vector bundle of rank r. Define $P \to M$ to be the frame bundle of E (so the fiber of P over $x \in M$ is the set of bases of the fiber of E over x). Then P is naturally a principal $\mathrm{GL}(\mathbb{R}^r)$-bundle and ∇ defines a horizontal bundle (Ehresmann connection) $\mathcal{H}$ by those linearized frames s at $p \in P$ with ∇s vanishing at p.

Example 7. *If $E = M \times \mathbb{R}^r$ is trivial, then $P = M \times \mathrm{GL}(\mathbb{R}^r)$ is trivial too. If ∇ is any connection on $M \times \mathbb{R}^r$, then we can write $\nabla = d + \eta$ where the connection 1-form $\eta \in \Omega^1(M, \mathrm{End}(\mathbb{R}^r))$ is a matrix-valued 1-form on M; that is, it is a map $\eta : TM \to \mathrm{End}(\mathbb{R}^r) = \mathfrak{gl}(\mathbb{R}^r)$. In the principal bundle setting, this corresponds to the 1-form*

$$A_{\mathrm{triv}} + \pi^*\eta \in \Omega^1(P, \mathfrak{gl}(\mathbb{R}^r))$$

on P, the kernel of which is the Ehresmann connection $\mathcal{H}$.

The takeaway is that, of the many different notions of "connection" that I have defined above, they are all equivalent in an appropriate sense.

A.7 Curvature

Suppose $A \in \Omega^1(P, \mathfrak{g})$ is a connection on a principal G-bundle $P \to M$. Define

$$\tilde{F}_A := dA + [A \wedge A] \in \Omega^2(P, \mathfrak{g}).$$

This is a basic 2-form, and so there is a unique $F_A \in \Omega^2(M, P(\mathfrak{g}))$ so that

$$\pi^* F_A = \tilde{F}_A.$$

This 2-form F_A is called the *curvature* of A. Note that if $\mathfrak{g}$ is abelian, then its Lie bracket vanishes and so $\tilde{F}_A = dA$.