# A note on product manifolds and shrinking á la Kaluza–Klein

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Suppose $Z = X \times Y$ is a product of two manifolds $X$ and $Y$, and equip $Z$ with a product metric. In this note I compute various metric quantities (e.g., the Hodge star and curvature) on $Z$ in terms of analogous quantities on $X$ and $Y$. My primary interest is to write down explicit expressions for how these quantities vary when the metric on the second factor is conformally scaled. These types of metric variations have been explored for a very long time. A now-famous instance of this appeared about a century ago in the physics literature: Kaluza [4] and Klein [5] observe that on a product 5-manifold $X^4 \times S^1$, when the $S^1$-factor is scaled to zero, the Einstein equation on $X^4 \times S^1$ decouples to produce the Einstein equation on $X^4$ *and* Maxwell's equation on $X^4$. My own interest in metric variations of this type are rooted in Atiyah's heuristic [1] motivating the Atiyah–Floer conjecture [2, 3]. Despite the fancy-sounding words of the previous several sentences, this note contains nothing particularly new or really even terribly interesting; it is just a collection of notes that I have found useful over the years and I figure they may be worth sharing.

In various examples below I will restrict to the case where $X = \mathbb{R}$ or $X = \mathbb{R} \times \mathbb{R}$; in both cases $X$ will be equipped with the standard metric and orientation. The variable $s$ (resp. ordered pair $(s, t)$) will always be the standard coordinate variable on $\mathbb{R}$ (resp. $\mathbb{R} \times \mathbb{R}$). For example, the metric on $\mathbb{R} \times \mathbb{R}$ is then $ds^2 + dt^2$, and the volume form is $ds \wedge dt$. If you, the reader, prefer compact spaces, you are free to replace $\mathbb{R}$ by $S^1$ at any time.

## 1 Basic Riemannian geometry of $(Z, g_\epsilon)$

### 1.1 Metrics

Throughout this document, $g_X$ (resp. $g_Y$) will be a metric on $X$ (resp. $Y$). The aforementioned product metric on $Z$ is the metric given by

$$g = \pi_X^* g_X + \pi_Y^* g_Y$$

on $Z$, where $\pi_X$ (resp. $\pi_Y$) is the projection to $X$ (resp. $Y$). I will refer to $g$ as the *standard metric* on $Z$.

These pullbacks in the definition of $g$ are cumbersome and their absence won't generally lead to a confusion, so I will generally drop them from the notation. For example, I will instead write things like

$$g = g_X + g_Y.$$

The specific metric variations I will consider are given by

$$g_\epsilon := g_X + \epsilon^2 g_Y$$

where $\epsilon > 0$ is a constant. Call $g_\epsilon$ the *$\epsilon$-metric* on $Z$.

**Remark 1.1.** *I like to think of $\epsilon$ as small, so $Z$ looks like a copy of $Y$ slightly "thickened" by $X$, but in principle this constant can be any value (e.g., very large). Of course, taking $\epsilon = 1$ this $\epsilon$-metric $g_1 = g$ recovers the standard metric on $Z$. This is the now-famous long wire analogy: from afar a long wire looks 1-dimensional, but if you zoom in it looks like a cylinder. The wire is $\mathbb{R} \times S^1$ and "from afar" corresponds to the metric on the $S^1$-factor being very small.*

## 1.2 Orientations

The orientations on $X$ and $Y$ induce an orientation on $Z = X \times Y$ via the "left-to-right convention". This is equivalent to postulating that the volume form $d\mathrm{vol}$ on $Z$ is the product

$$d\mathrm{vol} = d\mathrm{vol}_X \wedge d\mathrm{vol}_Y$$

of the (pullbacks of the) volume forms on $X$ and $Y$. The volume form just written down is the volume form of the standard metric. The volume form of the $\epsilon$-metric is

$$d\mathrm{vol}_\epsilon = \epsilon^{\dim(Y)} d\mathrm{vol}_X \wedge d\mathrm{vol}_Y = \epsilon^{\dim(Y)} d\mathrm{vol}.$$

Note that under the left-to-right orientation convention, the map

$$\begin{array}{ccc} X \times Y & \longrightarrow & Y \times X \\ (x, y) & \longmapsto & (y, x) \end{array} \tag{1}$$

is orientation-reversing if and only if $X$ and $Y$ are both odd-dimensional. This is because

$$d\mathrm{vol}_X \wedge d\mathrm{vol}_Y = (-1)^{\dim(X)\dim(Y)} d\mathrm{vol}_Y \wedge d\mathrm{vol}_X.$$

**Remark 1.2.** *If you are bothered by the fact that in this document I only consider the case where the metric on $Y$ in $X \times Y$ is scaled, leaving untreated the mirror case where the metric on $X$ is scaled, rest assured that you can use the diffeomorphism (1) to convert between the two cases and it will at worst cost you a minus sign (I will leave these details up to you).*

### 1.3 The pointwise norm

I will write $|\cdot|$ (resp. $|\cdot|_\epsilon$) for the pointwise norm relative to $g$ (resp. $g_\epsilon$), whether this is on vectors or on covectors. Then we have

$$\begin{aligned} |\text{vector on } X|_\epsilon &= |\text{vector on } X| \\ |\text{vector on } Y|_\epsilon &= \epsilon|\text{vector on } Y| \end{aligned}$$

where a "vector on $X$" is a vector on $Z = X \times Y$ that is supported in the $X$-directions (if you're confused by this, it would be a good exercise to define this rigorously). Viewing $\epsilon$ as small (less than 1, say) the above identities express the fact that vectors on $Y$ are getting smaller as we pass from $g$ to $g_\epsilon$ since if $v$ is a unit vector on $Y$ in the $g$-norm, then $v$ has length $\epsilon$ in the $g_\epsilon$-norm.

The analogous identities for covectors are obtained from duality:

$$\begin{aligned} |\text{covector on } X|_\epsilon &= |\text{covector on } X| \\ |\text{covector on } Y|_\epsilon &= \epsilon^{-1}|\text{covector on } Y|. \end{aligned}$$

Note that for $\epsilon < 1$, covectors (and hence 1-forms) are *larger* in the $\epsilon$-metric.

Following a standard procedure that is reviewed in the next section, each alternating product $\Lambda^k T^* Z$ inherits a fiber metric from $g_\epsilon$, and thus we obtain a pointwise metric and norm on $k$-forms. I will also write $|\cdot|_\epsilon$ for this norm and I will set $|\cdot| := |\cdot|_1$. Then this satisfies

$$\begin{aligned} |k\text{-form on } X|_\epsilon &= |k\text{-form on } X| \\ |k\text{-form on } Y|_\epsilon &= \epsilon^{-k}|k\text{-form on } Y|. \end{aligned} \tag{2}$$

**Remark 1.3.** *As discussed in the next section, there are actually several standard ways to define this inner product, and they have to do with whether one defines $\alpha \wedge \beta$ as $\alpha \otimes \beta - \beta \otimes \alpha$, or as this same formula but with a nonzero scalar (such as $1/\sqrt{2}$) in front. The identities (2) for $|\cdot|_\epsilon$ hold regardless of which choice of scalar one chooses.*

## 2 The Hodge star

Let $E$ be a real, oriented vector space of dimension $n$ and equip this with an inner product $\langle \cdot, \cdot \rangle$. Pick an oriented orthonormal basis $u_1, u_2, \ldots, u_n$ for $E$. Then the volume form associated to this inner product and orientation is $d\text{vol} := u_1 \wedge u_2 \wedge \ldots \wedge u_n$. The inner product on $E$ induces an inner product on the alternating product $\Lambda^k E$ for each integer $k$; this is reviewed in Section 2.4 for those interested. I will use the same symbol $\langle \cdot, \cdot \rangle$ to denote the inner product on $\Lambda^k E$, and I will write $|\cdot|$ for its associated norm.

The *Hodge star* on $E$ is the linear map $* : \Lambda^k E \to \Lambda^{n-k} E$ with the property that if $\alpha \in \Lambda^k E$, then $*\alpha \in \Lambda^{n-k} E$ is the unique multivector satisfying

$$\beta \wedge *\alpha = \langle \beta, \alpha \rangle d\text{vol}, \qquad \forall \beta \in \Lambda^k E. \tag{3}$$

If you haven't done this exercise, it is worth checking that $*\alpha$ really is uniquely determined by the condition (**??**). Moreover, since both sides of (**??**) are symmetric and bilinear in $(\alpha, \beta)$, one can show that the Hodge star is uniquely determined by the ostensibly weaker characterization that

$$\alpha \wedge *\alpha = |\alpha|^2 d\mathrm{vol}$$

for all $\alpha \in \Lambda^k E$ (this is the equivalence of a quadratic form and its associated symmetric bilinear form, but it too is a worthwhile computation if you haven't gone through it). In Section 2.5 I give yet another characterization of Hodge star, this time in terms of bases.

While you're at it, you may as well prove the following useful identities:

$$\begin{aligned} *^2\alpha := *(*\alpha) &= (-1)^{k(n-k)}\alpha, \qquad \forall \alpha \in \Lambda^k E \\ *(1) &= d\mathrm{vol}. \end{aligned}$$

Now suppose $M$ is an oriented manifold with a Riemannian metric. Then each cotangent space $T_p^* M$ is an oriented vector space with an inner product and so admits a Hodge star. This varies smoothly with the basepoint $p$ and so the pointwise Hodge star determines a bundle map

$$* : \Lambda^\bullet T^* M \longrightarrow \Lambda^\bullet T^* M$$

covering the identity. This bundle map is called the *Hodge star* on $M$. Applying the Hodge star on $M$ to the values of sections gives a linear map on forms (i.e., on sections of $\Lambda^\bullet T^* M$), and this map is also called the *Hodge star*.

## 2.1 Product manifolds

As described in the previous section, the metric and orientation on $X$ (resp. $Y$) induces a Hodge star $*_X$ on $X$ (resp. $*_Y$ on $Y$). Likewise, the manifold $Z$ has a Hodge star $*$ coming from the standard metric and orientation specified above. Let's set $n := \dim(Z)$. The question I want to answer is this:

*How can we compute $*$ in terms of $*_X$ and $*_Y$?*

Let's start with a warm-up: Let $\xi \in \Lambda^\bullet T^* X$ be of pure degree on $X$, but view it as a form on $Z$ (e.g., by pulling it back under the projection $Z = X \times Y \to X$). The key feature is that, since $g$ is a product metric, the norm of $\xi$ on $X$ equals its norm on $Z$. The definition of the Hodge star on $Z$ says $*\xi$ is uniquely determined by

$$\alpha \wedge *\xi = |\xi|^2 d\mathrm{vol} = |\xi|^2 d\mathrm{vol}_X \wedge d\mathrm{vol}_Y. \tag{4}$$

Likewise, the definition of the Hodge star on $X$ says $*_X\xi$ is uniquely determined by

$$\xi \wedge *_X \xi = |\xi|^2 d\mathrm{vol}_X.$$

Using this latter identity, we have

$$\xi \wedge (*_X \xi) \wedge d\mathrm{vol}_Y = |\xi|^2 d\mathrm{vol}_X \wedge d\mathrm{vol}_Y = |\xi|^2 d\mathrm{vol}$$

which shows that $(*_X\xi) \wedge d\mathrm{vol}_Y$ solves the identity (3) in place of $*\xi$. Since (3) uniquely characterized $*\xi$, this shows

$$*(\xi) = (*_X\xi) \wedge d\mathrm{vol}_Y.$$

Now let's do the analogous computation for $Y$: Fix $\psi \in \Lambda^\bullet T^*Y$ of pure degree. This time we need to reorder:

$$\begin{aligned}
\psi \wedge *\psi &= |\psi|^2 d\mathrm{vol} \\
&= |\psi|^2 d\mathrm{vol}_X \wedge d\mathrm{vol}_Y \\
&= d\mathrm{vol}_X \wedge \left(|\psi|^2 \psi \wedge *_Y\psi\right) \\
&= (-1)^{|\psi|\dim(X)} \psi \wedge d\mathrm{vol}_X \wedge *_Y\psi.
\end{aligned}$$

Here $|\psi|$ is the degree of $\psi$. From this, we conclude that

$$*\psi = (-1)^{|\psi|\dim(X)} d\mathrm{vol}_X \wedge *_Y\psi.$$

In general, the reader is invited to prove the following via similar lines.

**Proposition 2.1.** *If $\xi \in \Lambda^k T^*X$ and $\psi \in \Lambda^\ell T^*Y$, then*

$$*(\xi \wedge \psi) = (-1)^{\ell(\dim(X)-k)} (*_X\xi) \wedge (*_Y\psi).$$

The reason there is an asymmetry in $k = |\xi|$ and $\ell = |\psi|$ in the power of $-1$ is due to our left-to-right convention which 'prefers' one over the other.

As an example, when $X = \mathbb{R}$ and $\psi \in \Lambda^\ell T^*Y$ we have

$$\begin{aligned}
*\psi &= (-1)^\ell dt \wedge *_Y\psi \\
*dt &= d\mathrm{vol}_Y \\
*(dt \wedge \psi) &= *_Y\psi.
\end{aligned}$$

since $\dim(\mathbb{R}) = 1$.

As another example, when $X = \mathbb{R} \times I$ and $\psi \in \Lambda^\ell T^*Y$, we have

$$\begin{aligned}
*ds &= dt \wedge d\mathrm{vol}_Y \\
*dt &= -ds \wedge d\mathrm{vol}_Y \\
*\psi &= ds \wedge dt \wedge *_Y\psi \\
*(ds \wedge dt) &= d\mathrm{vol}_Y \\
*(ds \wedge \psi) &= (-1)^\ell dt \wedge *_Y\psi \\
*(dt \wedge \psi) &= (-1)^{\ell+1} ds \wedge *_Y\psi \\
*(ds \wedge dt \wedge \psi) &= *_Y\psi.
\end{aligned}$$

## 2.2 Conformal scaling

Let me begin with the following general observations about conformal scaling: Suppose $G$ be any metric on a manifold $M$, and let $c > 0$ be a scalar. Writing $|\cdot|_G$ and $d\text{vol}_G$ for the norm and volume form associated to $G$, we have

$$\begin{aligned} |\text{vector on } M|_{c^2G} &= c|\text{vector on } M|_G \\ |k\text{-form on } M|_{c^2G} &= c^{-k}|k\text{-form on } M|_G \\ d\text{vol}_{c^2G} &= c^{\dim(M)} d\text{vol}_G. \end{aligned}$$

Now suppose that $\mu$ is a form on $M$ of pure degree $|\mu|$, and $*_c$ is the Hodge star associated to $c^2G$. Setting $* = *_1$ we have

$$*_c\mu = c^{\dim(M)-2|\mu|} * \mu.$$

This shows how the Hodge star behaves relative to conformal rescaling. This shows that the Hodge star is conformally invariant in the middle dimension (odd-dimensional manifolds don't have a middle dimension).

## 2.3 The $\epsilon$-metric

Next, let $*_\epsilon$ be the Hodge star on $Z$ associated to the $\epsilon$-metric $g_\epsilon = g_X + \epsilon^2 g_Y$. Then we can immediately deduce the following from Proposition 2.1 and the above conformal scaling properties.

**Corollary 2.2.** *If $\xi \in \Lambda^k T^*X$ and $\psi \in \Lambda^\ell T^*Y$, then*

$$*_\epsilon(\xi \wedge \psi) = (-1)^{\ell(\dim(X)-k)} \epsilon^{\dim(Y)-2\ell} (*_X\xi) \wedge (*_Y\psi).$$

Thus we have a formula for $*_\epsilon$ that shows the explicit dependence of this quantity on $\epsilon$.

## 2.4 Appendix 1: Inner products on tensors and wedges and things

The point of this section is to give an overview of how the inner product on a vector space $E$ induces on one $\Lambda^k E$ for each $k$. If you are already happy with this fact, feel free to skip this section (frankly, you can skip this whole dang paper if you want!).

Let $E_1, E_2$ be vector spaces. Assume $E_j$ is equipped with an inner product $\langle \cdot, \cdot \rangle_{E_j}$, and write $|\cdot|_{E_j}$ for its associated norm. Then these induce an inner product on $E_1 \otimes E_2$ by multiplying

$$\langle e_1 \otimes e_2, e_1' \otimes e_2' \rangle_{E_1 \otimes E_2} := \langle e_1, e_1' \rangle_{E_1} \langle e_2, e_2' \rangle_{E_2}.$$

This induces the norm

$$|e_1 \otimes e_2|_{E_1 \otimes E_2} := |e_1|_{E_1} \, |e_2|_{E_2} \, .$$

Similarly, we can define an inner product on $E_1 \oplus E_2$ by adding

$$\langle e_1 \oplus e_2, e_1' \oplus e_2' \rangle_{E_1 \oplus E_2} := \langle e_1, e_1' \rangle_{E_1} + \langle e_2, e_2' \rangle_{E_2}.$$

This induces a norm $|\cdot|_{E_1 \oplus E_2}$ that satisfies

$$|e_1 \oplus e_2|^2_{E_1 \oplus E_2} = |e_1|^2_{E_1} + |e_2|^2_{E_2} \, .$$

As for alternating products, view

$$E_1 \wedge E_2 \subset (E_1 \otimes E_2) \oplus (E_2 \otimes E_1)$$

as the subbundle generated by terms of the form $e_1 \wedge e_2 := \frac{1}{\sqrt{2}}(e_1 \otimes e_2 + e_2 \otimes e_1)$. Then by the above this inherits a metric and a norm. For example, the norm is given by

$$\begin{aligned} |e_1 \wedge e_2|^2_{E_1 \wedge E_2} &= \tfrac{1}{2} \left( |e_1 \otimes e_2|^2_{E_1 \otimes E_2} + |e_2 \otimes e_1|^2_{E_2 \otimes E_1} \right) \\ &= |e_1|^2_{E_1} \, |e_2|^2_{E_2} \end{aligned} \tag{5}$$

(Note that I have used a not-necessarily-standard convention by including a $\sqrt{2}$ in the definition of the wedge product; otherwise the right side of (4) would have a factor of 2 on the right.)

Now consider the case where $E_1 = E_2$ are equal, and set $E := E_1$. There is another way to arrive at this inner product on $E \wedge E$: Fix an orthonormal basis $u_1, \ldots, u_n$ for $E$. Then $u_i \wedge u_j$ for $1 \leq i < j \leq n$ is a basis for $E \wedge E$, and the inner product defined in the previous paragraph is the unique one for which this frame is orthonormal.

The above discussion extends to products, sums, and alternating products of more than two vector spaces as well, though there is a choice involved for the wedge product (analogous to my choice of using a $\sqrt{2}$). I will pin this down but only in the special case I need, which is the case in which all vector spaces in sight are equal. Specifically, using the notation of the previous paragraph, on the alternating product $\Lambda^k E$ I will consider the inner product for which the frame

$$u_{i_1} \wedge u_{i_2} \wedge \ldots \wedge u_{i_k}, \qquad \text{for } 1 \leq i_1 < i_2 < \ldots < i_k \leq n$$

is orthonormal. Then this satisfies

$$|\alpha \wedge \beta| = |\alpha||\beta|$$

for any

$$\alpha, \beta \in \Lambda^\bullet E := \bigoplus_k \Lambda^k E.$$

### 2.5 Appendix 2: Basis characterization of $*$

Here I will outline another equivalent characterization of the Hodge star that can be useful at times (though I prefer to avoid it if I can). More specifically, the Hodge star can be defined in terms of the frame for $\Lambda^k E$ coming from a choice of orthonormal basis $u_1, \ldots, u_n$ for $E$. To describe this, it is convenient to use multi-index notation here:

$$u_I := u_{i_1} \wedge u_{i_2} \wedge \ldots \wedge u_{i_k}$$

where $I := (i_1, i_2, \ldots, i_k)$ is an ordered tuple of $k$ integers $i_1, \ldots, i_k \in \{1, \ldots, n\}$. Unless otherwise specified, I will always assume the entries of $I$ are strictly increasing. Note that an orthonormal basis for $\Lambda^k E$ is given by the set $u_I$ for strictly increasing multi-indices $I = (i_1, i_2, \ldots, i_k)$ of length $k$.

Define $I^c$ to be the multi-index of length $n-k$ obtained by listing the complementary indices $\{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$ in increasing order. With some careful thought, but without writing anything down, I was able to convince myself that there is some $\sigma_I \in \{-1, 1\}$ with

$$u_I \wedge u_{I^c} = \sigma_I d\text{vol}.$$

I bet you can convince yourself too! If you want a bit more, here are some extra words: the sign is determined by counting the interchanges necessary to arrange the entries of $u_I \wedge u_{I^c}$ so the indices are in increasing order.

With this notation in hand, the Hodge star is equivalently characterized by the identity

$$*e_I = \sigma_I e_{I^c}$$

for all multi-indices $I$. Since the $e_I$ are a basis for $\bigoplus_k \Lambda^k E$, this can be used to define $*$ uniquely.

# 3 Curvature

### 3.1 The Riemannian Curvature Tensor

In general, for a metric $G$ on a manifold $M$, we have the Riemannian curvature endomorphism $R^G$ defined by

$$\begin{aligned} R^G : TM \otimes TM \otimes TM &\longrightarrow TM \\ (X, YZ) &\longmapsto R^G(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \end{aligned}$$

where $\nabla$ is the Levi-Civita connection. The Riemmanian curvature tensor is then defined by

$$Rm^G(W, X, Y, Z) := G(W, R^G(X,Y)Z).$$

This Riemannian curvature tensor is well-behaved under products in the following sense: If $M = M_1 \times M_2$ and $G = G_1 + G_2$ is a product, then

$$\begin{aligned} &Rm^G(W_1 + W_2, X_1 + X_2, Y_1 + Y_2, Z_1 + Z_2) \\ &\quad = Rm^{G_1}(W_1, X_1, Y_1, Z_1) + Rm^{G_2}(W_2, X_2, Y_2, Z_2), \end{aligned}$$

where $W_i, X_i, Y_i, Z_i$ are tangent vectors on $M_i$. That is,

$$Rm^G = Rm^{G_1} + Rm^{G_2}$$

where it is understood that the tensors on the right are pulled back to $M_1 \times M_2$ via the obvious projections.

We need one more formula: Suppose $c > 0$ is a constant. The associated Riemannian curvature tensor of $c^2G$ is related to that of $G$ in the expected way:

$$Rm^{c^2G} = c^2 Rm^G.$$

With these tools in hand, let's now turn to the $\epsilon$-metric $g_\epsilon = g_X + \epsilon^2 g_Y$ on $Z = X \times Y$. Define

$$Rm^\epsilon := Rm^{g_\epsilon},$$

and set $Rm^X := Rm^{g_X}$ and $Rm^Y := Rm^{g_Y}$. Then applying the above formulas, we obtain

$$Rm^\epsilon = Rm^X + \epsilon^2 Rm^Y.$$

For example, this shows that if $X$ is flat, then $Rm^\epsilon = \epsilon^2 Rm^Y$, while if $Y$ is flat, then $Rm^\epsilon = Rm^X$ is independent of $\epsilon$.

The Ricci curvature satisfies the same sort of product property that the Riemannian curvature tensor satisfies:

$$\mathrm{Ric}^G = \mathrm{Ric}^{G_1} + \mathrm{Ric}^{G_2},$$

However, it is invariant under constant conformal rescaling:

$$\mathrm{Ric}^{c^2G} = \mathrm{Ric}^G$$

($c > 0$ needs to be a *constant* here). This implies that the Ricci curvature of $g_\epsilon$ is independent of $\epsilon$:

$$\mathrm{Ric}^\epsilon = \mathrm{Ric}^X + \mathrm{Ric}^Y,$$

where I hope by now you can guess at what I mean by $\mathrm{Ric}^\epsilon$, $\mathrm{Ric}^X$, and $\mathrm{Ric}^Y$.

# References


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