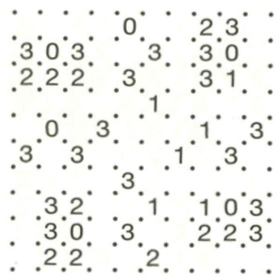
MATH 103 OVERVIEW

Topics:
Japanese Pencil puzzles (Slitherlink)
Finite Geometries
Perspective Drawing
Symmetries

Japanese Puzzles (Slitherlink)

A Slitherlink puzzle looks like:

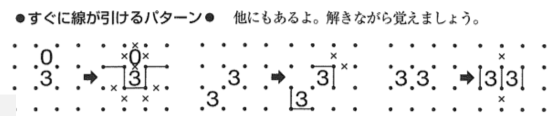


The point of a Slitherlink is to create a simple closed curve (one that starts and ends in the same place and is one continuous arc with no branching) which includes the number of sides indicated. For example, for every 3, 3 of the four sides surrounding that 3 are part of the curve, and 0 of the four sides surrounding a 0 are part of the curve.

Some of the consequences of those simple rules (theorems) are

Every 3 looks like one of the following: 

Here are some hints about 3s provided by Nikoli, the puzzle company who invented Slitherlink puzzles:



Other observations include: three options for 0 and 2 next to each other

There is a deep mathematical theorem that can help us solve a Slitherlink (especially the difficult ones) which is

Theorem 1. Jordan Curve Theorem *A simple closed curve has an inside and an outside.*

This theorem tells us that we can color in the inside of any genuine solution to a Slitherlink, even very, very large ones, and this can sometimes help us when none of our other theorems are applicable.

Finite Geometries

A Finite Geometry is a finite collection of points and lines with the rule that

- Every pair of points is connected by exactly one line.

It's a good idea to keep in mind that here, lines do not need to be straight, they can curve or even loop back on themselves to make a circle. For example, there are two different finite geometries with 3 points: one with three distinct lines and one where all three points lie on the same line. To make things easier to read, we try to draw our lines without sharp corners and try to make different lines different colors.

One of the best parts of Finite Geometries is that we can prove things by drawing pictures!

To make the geometries more interesting, we added a second rule

- There exist four points no three of which are on the same line.

One of the most important aspects of a finite geometry is how it deals with parallel lines (that is, do they exist, and if they exist, are they unique). Each of these three options (exits and are unique, exist and are not unique, and don't exist) produces a different type of finite geometry. These are Affine Geometries, Hyperbolic Geometries, and Projective Geometries respectively. The first of these, both historically and in our class, is the Finite Affine Geometries.

A Finite Affine Geometry is a collection of finitely many points and lines such that

- Every pair of points is connected by exactly one line.
- There exist four points no three of which are on the same line.
- If there is a line and a point that is not on that line, there must also be exactly one line that runs through that point and is parallel to the first line.

It should be noted that the definition of parallel lines for a finite geometry is lines that do not meet. The lines do not need to look parallel!

This last one is sometimes known as Playfair's Axiom and it paraphrases to 'parallel lines always exist and are unique'.

Although the rules for a finite Finite Affine Geometry seem simple and not too restrictive, they create a surprising amount of structure. For example, we proved in an in-class activity that there are the same number of points on every line! This depends on the much easier and not terribly surprising fact that for every pair of lines, there is a point not on either of them (it's a direct consequence of the second rule). Once we've found that other point, every point on the first line has a line between itself and that point P. At most one of these is parallel to the second line, so these can be used to associate to each point on the first line a unique point on the second line. The same argument in the other direction (switching which line is 'first') shows that the two lines have the same number of points!

We studied the smallest Finite Affine Geometry (four points and six lines) as well as the next smallest with nine points and twelve lines.

The next sub-type of Finite Geometry we studied is the Finite Projective Geometry which has the same first two rules as a Finite Affine Geometry. The third rule is replaced in the following way:

- Every pair of points is connected by exactly one line.
- There exist four points no three of which are on the same line.
- Every pair of lines meet at exactly one point.

Namely, this is the type of geometry for whom parallel lines do not exist! Because of this rule, a four point Finite Projective Geometry doesn't exist, but a seven point one does, and it even has a name! It's called the Fano Plane.

The fact that every line has the same number of points on it is easier to prove for a Finite Projective Geometry and we did this in class. It's also true that any Finite Affine Geometry can be turned into a Finite Projective Geometry by adding a 'point at infinity' for each set of parallel lines and a 'line at infinity' connecting up these points. Similarly, a Finite Projective Geometry can be turned into a finite Affine Geometry by removing one line and all of its points.

In fact, the first and third rule of a Finite Projective Geometry are opposites with the role of points and lines switched. This tells us that the number of points on each line is equal to the number of lines through each point (which was not the case with Affine Geometries).

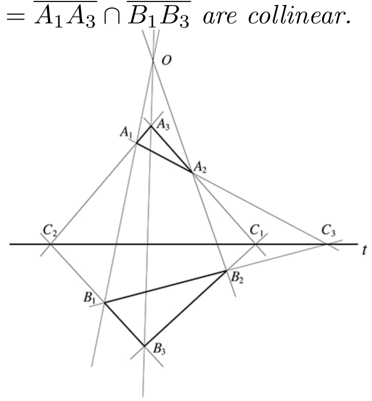
The final type of Finite Geometry we studied is a Finite Hyperbolic Geometry. Here, parallel lines exist but are not unique, so our third rule becomes

- If there is a line and a point that is not on that line, there must also be more than one line that runs through that point and is parallel to the first line.

Such a geometry can be modeled on a potato chip or using the Poincaré disk model of the hyperbolic plane where the entire plane lives inside a circle and 'lines' in the plane are pieces of circles which meet the boundary circle at right angles. We drew some examples of Finite Hyperbolic Geometries on the Poincaré disk, so we proved that they exist. However, we found that more concrete explorations, like our proof that every line has the same number of points on it (called the *order* of a Finite Geometry) fail utterly for Hyperbolic Geometries.

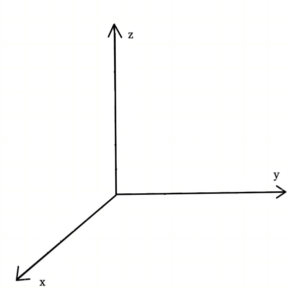
The last topic we covered in Finite Geometries was a Theorem about perspective drawing:

Theorem 2 (Desargues)**.** *The vertices of two triangles $\triangle A_1A_2A_3$ and $\triangle B_1B_2B_3$ are in perspective with respect to a point O if and only if the edges of the two triangles are in perspective with respect to the line t. That is, the lines $\overline{A_1B_1}, \overline{A_2B_2}$ and $\overline{A_3B_3}$ intersect at point O if and only if the points $C_1 = \overline{A_1A_2} \cap \overline{B_1B_2}$, $C_2 = \overline{A_2A_3} \cap \overline{B_2B_3}$ and $C_3 = \overline{A_1A_3} \cap \overline{B_1B_3}$ are collinear.*



The context for this theorem is Finite Projective Geometries (there are no parallel lines here), but surprisingly, the theorem is independent of the axioms. In other words, there exist Finite Projective Geometries where Desargues's Theorem does not hold.

Desargues's theorem was our transition into...



Perspective Drawing

We started this topic by talking about how to mathematicians diagonally draw in three dimensions, with an x, y and z axis. The standard drawing (shown above) has the x axis pointing left and diagonally down, to imply it is coming out of the paper, while the y axis points right and the z axis points up. We modeled this with a study carrel. The three most important planes are the coordinate planes: the xy plane (the desk plane), the yz plane (the study carrel plane) and the xz plane (the wall to the left of the study carrel).

When we drew the coordinate planes, and planes that are parallel to the coordinate planes, we observed that the edges of the planes were always parallel to one of the three coordinate axes. Every pair of planes either meet in a line or are parallel to each other (and this time they do look 'parallel').

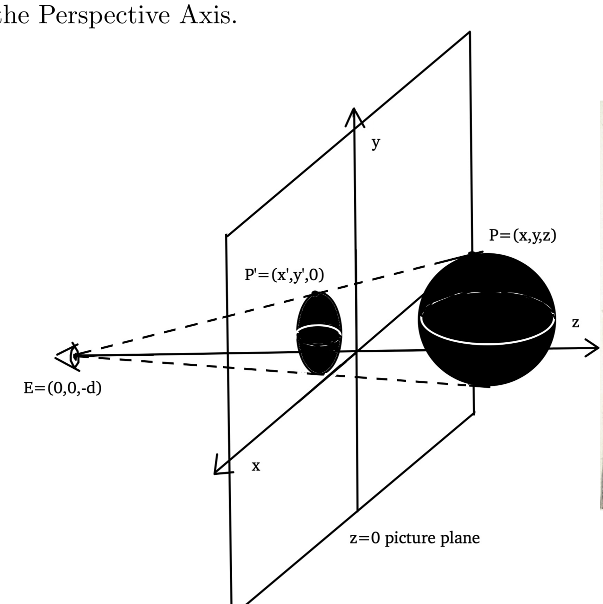
In ordinary graphing with the xy plane in 2 dimensions, the axes are marked with equally spaced intervals. However, when we draw in 3 dimensions, the marks on the x axis appear to be closer together than those on the y and z axis due to foreshortening (things look shorter when viewed from an angle). Points in 3space are located by using the tick marks on the axes to create boxes of the appropriate size. One corner of the box is always located at the origin (the point $(0, 0, 0)$). The point opposite the origin is the point we were trying to graph! Negative coordinates are indicated by having the box go backward or leftward or downward instead of forward, rightward, and upward, as we use for positive coordinates.

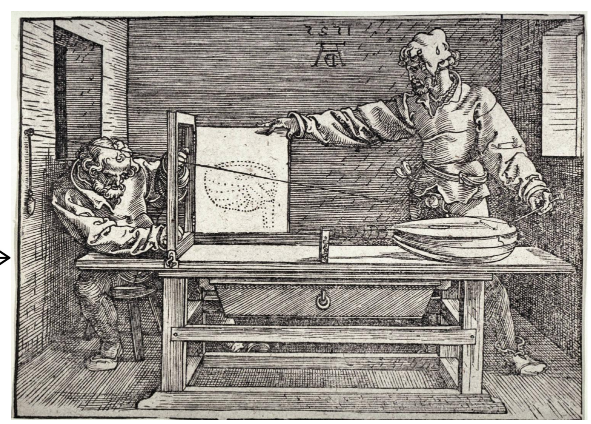
The other thing we learned to draw in mathematicians 3space are boxes. In particular, we drew boxes that look like the ones in the previous paragraph, but can be located anywhere in space. Each of the points on these boxes has a box of its own to locate it, but we usually only draw the location boxes for the four points nearest the origin so that our pictures are readable. The locations of the rest of the points on these more general boxes can be deduced graphically from those four points.

There are many interesting things about this mathematician's version of 3dimensional space. One is that since it is a drawing, it loses information! So there are multiple points different points in 3 space that have to share the same spot on the xyz axes' picture. To show which point we mean, we need to draw its location box! Another interesting thing is that mathematicians use a 'parallel projection' where parallel lines in 3space look parallel in our drawings. This runs somewhat counter to our lived experience of the world. In our everyday experience, very long parallel lines appear to meet at a 'vanishing point'.

Because mathematical drawings do not use a vanishing point, they can look less realistic. But they are more useful for planning real things! The mathematicians xyz space is excellent for architecture and technical drawings as it keeps parallel lines looking parallel.

Mathematical perspective drawing, which does have vanishing points, involves using a special set up that we called the Perspective Axis.



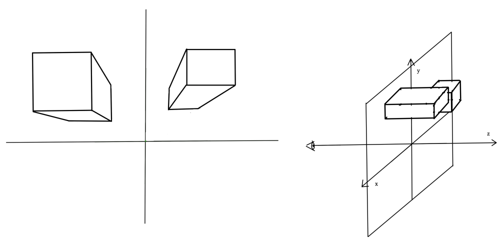


In this drawing method, the x axis faces outward as usual, but this time the y axis goes up and the z axis goes to the right. This is so that the x and y axes form the xy plane which is the picture plane. There is an eyeball located at the point $(0, 0, -d)$ leftward on the z axis, and what is drawn on the picture plane is from the point of view of that eyeball. That is, if there is an object located to the right of the picture plane, it is drawn by drawing in sight lines from the eyeball to the object and marking where these sight lines intersect the picture plane. This is just like the mechanical device that Dürer pictured at the right, where the rôle of the eyeball is played by the nail on the wall.

This method of perspective drawing (sight lines intersecting the picture plane) can be turned into equations using algebra and 'similar triangles'. We first used these equations to compute x' and y' coordinates for a point with constant x and y coordinates that was moving away from the eyeball. We found that the farther away this point got (larger z coordinates), the closer the image of the point in the picture plane got to the origin (the point $(0, 0)$). This hinted at the existence of vanishing points. Using this method, these equations can also be used to graph simple objects like boxes in perspective. This is how we completed our First Perspective Drawing (we computed x' and y' for all eight corners of a box). We found that our box also had a vanishing point, and we explored the fact that all sets of parallel lines have their own vanishing point, which is found by intersecting the sight line parallel to those lines with the picture plane. This means that drawings in mathematical perspective not only have a vanishing point, but have multiple vanishing points, one for each set of parallel lines!

Our next topic was one-point perspective. In one-point perspective, unlike the full mathematical perspective described above, there is only one vanishing point. Only the lines that are parallel to the z axis vanishing. This corresponds to lines that are perpendicular to the picture plane having a vanishing point of $(0, 0)$ in the middle of the picture plane. It is also possible to see one-point perspective in real life! Because man-made objects (like streets and buildings) are mostly made out of rectangles. For example, in a rectangular room, the ceiling, the floor and the side walls all have the same vanishing point, as they are all parallel to each other. If the viewer/camera/eyeball is on the front wall, then that vanishing point is directly opposite. We saw this in photographs by drawing on them all of the parallels to the vanishing point (my samples were of the Merchandise Mart in Chicago and of a street in Manhattan, both of whom had a vanishing point in the middle of the photograph, but there were other options as well). We also used a perspective photo to figure out what the original space would have looked like in our parallel projection, like with the axis above on the left.

In addition to being able to draw a perspective drawing by computing its corners, we also learned how to draw boxes in perspective free hand, and how to recreate those boxes in parallel perspective.



(Notice the eyeball on the left side of the right image, that eyeball indicates that we are in our -mathematician's - parallel projection).

Our next topic was viewing distance. We discovered that how we see the box is greatly affected by the distance a box is viewed from. Even a box whose depth is twice its length can look like a (very blurry) cube if it is viewed through one eye from a close enough distance. We used similar triangles to see that if we are looking at a cube, the ideal viewing distance (from which only our cubes look like cubes and the whole drawing looks maximally 3dimensional) is the same as the distance between the vanishing point of a diagonal of the cube (along the horizon line) and the vanishing point of the whole picture. For a box twice as long as it is wide, the ideal viewing distance is twice that dimension. We were later able to use these facts to help us draw a perspective picture of a house using its 2D drawings.

Our last topic in one-point perspective drawing was regularly spaced patterns, There is an algorithm for creating a duplicate of a rectangle in 2D and this process still works in perspective, giving us the correct amount of space between objects even though far away objects appear smaller in a perspective drawing.

For a final wrap-up for the Perspective Drawing topic, we created a perspective drawing of a house using top and side views and dimensions. There were 11 basic moves that we could make for drawing the house given one of its front corners and the vanishing point, but in order to draw the rest of the house, we needed to know how far away from the vanishing point the ideal viewing distance was. We need to know this because what a square looks like depends on the viewing distance. We also needed to use the repeated pattern technique to draw the back of the house.

Symmetry

Our last topic was symmetry, and although it only took up two classes, it appears to be going quite well with its structure of homework problems and reference reading.

The first sub-topic in Symmetry was finding the symmetries of designs in the plane. There are only four types of rigid motion that preserve a plane. They are rotation around a point, reflection over a line, translation (picking up and moving the plane to a different place), and glide reflection which is a combination of a translation and a reflection across a line parallel to the direction of translation. For a finite design translation and glide reflection aren't possibilities as a figure preserved under these must be infinite.

For Homework 4.1, we were given a bunch of designs and had to decide if the figures were preserved under (aka look the same after performing) reflections and or rotations. The symmetry type of the figure is D_n if the figure can be reflected over n different lines and still look the same, and it is C_n if it can be rotated an angle of $\frac{360}{n}$ degrees and still look the same. Notice that every figure has at least C_1 symmetries as rotation by 360 degrees preserves everything! If a figure has 2 or more reflections, it comes with rotations for free, but the D_n type symmetry is a stronger statement, so we usually don't bother writing the C_m in these cases.

Homework 4.2 is about friezes which are repeated patterns that extend infinitely to the left and the right. We will call the line that the frieze lies on its axis. There are exactly seven classes of friezes and just with like our finite figures, we classify them by their symmetries. The only options for rigid motions that preserve a frieze are Translation, Half turns, $R_T$eflections perpendicular ('transverse') to the axis, $R_L$eflection over the axis of the frieze (longitudinal), and Glide reflections over the axis. The seven types of friezes are all preserved by some combination of these moves, where the easiest have just translation and possibly one other move (T, TH, TR_T, TR_L, TG) and the other two have multiple moves. TRG has translation, R_T and G as well as H, and TR^2 is invariant under all the moves. The main reason that there are only seven of these is the fact that any combination of two or more of the above rigid motions must be the same as a single rigid motion. Later in Homework 4.3 and Homework 4.4 and in the handout, there are some ways to figure out what the product of two or more motions are, but we're unlikely to get to that because of our digression into Wallpaper Patterns!

Homework 4.2b is about the notation we can use to decorate our figures that help us indicate the symmetries. The fundamental region of a figure is the smallest part of the figure necessary to produce the whole figure, and they are indicated by outlines or little bits of graph paper. A rotation of angle $\frac{360}{n}$ degrees is indicated by an n-gon on the center of the axis of rotation (so a rotation of 90 degrees is indicated by a square) with 180 degrees being indicated by a diamond (there is no $2 - gon$). Any axis of reflection is indicated by a solid line where multiple colored lines in a figure indicate multiple different reflections. Glide reflections are indicated by a dotted line along their axis and translations by an arrow giving the direction and length of the translation. Homework 4.2b asks you to decorate your figures from Homeworks 4.1 and 4.2 with our new notation.

Homework 4.2c is about creating your own friezes. For each of the seven frieze types, you are given an empty fundamental region as well as the symmetries of the frieze and are asked to create a frieze of your own and name it as one of the T, TH, etc..

Homework 4.2d is about the generalization of friezes to two dimensions, which are called wallpaper patterns. There are exactly 17 of these (much harder to show than the seven frieze patterns) and 8 of those are given with their blank fundamental region and symmetries for you to create patterns of that type.

Notice that the problems for your group-of-one final assignment will only cover through Homework 4.2d.

Homework 4.2e is just like 4.2d but with all of the 17 patterns. Even though they are just following directions (take your design and reflect/rotate/glide reflect), they can be surprisingly difficult. I made a mistake twice with the answer key for the first one on the second page!

Homework 4.3 is about Direct and Opposite motions and fixed points. Direct motions are rigid motions like translation and rotation that don't switch sides of the paper. Opposite motions are rigid motions like reflection and glide reflections that switch which side of the paper we are looking at. Fixed points are points that don't change under a motion, like the center of a rotation or the axis of a reflection) This exploration is headed towards simplifying the composition of multiple rigid motions to one of our four possibilities.

Homework 4.4 where you explore what single rigid motion the composition of two or more motions is. We know it's one of them because there are only 4 possibilities, but unlike things like a translation followed by a reflection on a access parallel to the translation being a glide reflection, most compositions are not so obvious!

Homework 4.5 is about how the set of symmetries of a figure like a square behave under composition. When you have a finite number of symmetries, you can make a chart and cover all of the compositions!

There are answer keys to all of these assignments available after you hand in the assignment (you can modify your assignment using the answer key, but you must include a sentence about either why you were wrong or why the correct answer is correct).

Here is a complete list of the miro boards that we used. To study for your final group-of-one assignment, look over these boards and make sure you understand what you group did. If you understand what happened on these boards, you will do great!

- 'Defining Sandwiches' (making sure our mathematical definitions are precise enough to account for everything we want them to do)
- 'Japanese Pencil Puzzles Days 1&2' (the board for section 8 contains a translation of the rules of Slitherlink!)
- 'Slitherlink Day 3 (Theorems)' is where you can find a list of the theorems we discovered while doing slitherlink puzzles as well as the start of our exploration of the Jordan Curve Theorem (there is a video of its proof linked under Pencil Puzzle Class Videos)
- Finite Geometries (all 9 activities discussed above are on this board)
- 'perspective drawing' or '[RESTORED] perspective drawing' contains our first four activities for perspective drawing (drawing planes, drawing boxes, perspective spheres, and perspective dinosaurs)
- 'Boxes and Planes' is our group-of-one assignment. This covered plotting points, drawing planes, drawing boxes, and overlapping points.
- 'Perspective Drawing Part II' contains our next four perspective drawing activities (the Algebra of Perspective, Moving Point, First Perspective Drawing, Vanishing Points)
- 'One-Point Perspective and Viewing Distance' contains our next four activities (analyzing paintings and photos, drawing boxes, viewing distance, and repeated patterns)
- 'House' (contains the drawings we used to construct our house in Geogebra)
- 'Symmetry Boards', there is one board for each group. It contains the Symmetry Handout and Homeworks 1,2,2b,2c,2d,3,4,5 (3,4,5 are not covered on the final assignment).