238 Chapter 2 Test

February 19, 2013 10/ **Name:** KEY 25

By printing my name I pledge to uphold the honor code.

The last page of this exam is a pair of matrices and their reduced row echelon form. They will help with the computational portion of the exam (in more than one place). It is possible you will not need all of the matrices on the back page and it is also possible that you will want the reduced row echelon form of a matrix not on this list. In some cases, the reduced row echelon form of a matrix is not provided because there are 'tricks' that can be used to tell us enough information about the rref that we don't need to actually compute it.

1. Fill in the blank.

4 4 **a)** The equation we solve to see if $\{v_1, v_2, ..., v_n\}$ are linearly independent/dependent is

$c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ (solve for c_i).

The vectors are linearly independent if $c_1 = c_2 = \cdots = c_n = 0$ only solution.

2 **b)** The Span of the set $\{v_1, v_2, ..., v_n\}$ is (give an English/mathematical expression that will be useful for the computation section of the exam)

$\{ c_1v_1 + c_2v_2 + \cdots + c_nv_n$ where $c_i \in \mathbb{R} \}$.

4 **c)** A basis for a vector space V is a subset $\mathcal{B}$ such that the elements of $\mathcal{B}$ are

linearly independent and span V.

2. Which of the following are vector spaces and **why/why not**? (You don't need to show all of the properties if a set is a vector space, just indicate why they hold or which ones fail to hold.)

a) $\mathbb{R}^{\geq 0}$ under the operations of ordinary addition and ordinary multiplication.

5 NOT A VS, INVERSES DON'T EXIST. $2+(-2)=0$ BUT $-2 \notin \mathbb{R}^{\geq 0}$.

YES, BUT NOT A SUBSPACE 2 OR 3

$\mathbb{R}^0, \mathbb{R}^1, \mathbb{R}^2, \ldots$ 4 (SINCE $\mathbb{R}^2, \ldots$ ISN'T UNDER ORDINARY +)

3. Which of the following are subspaces of the given vector space and **why/why not?**

2 PTS FOR CORRECT ANSWER.

a) The subset of $M_{2\times 2}(\mathbb{R})$ consisting of invertable matrices.

5 NOT A SUBSPACE, NOT CLOSED UNDER S. MULT. $0\begin{pmatrix}1&0\\0&1\end{pmatrix} = \begin{pmatrix}0&0\\0&0\end{pmatrix}$ & NOT INVERTABLE.

STATED BUT NO EXAMPLE (4 PTS)

b) Let the vector space $V = F[0,1]$ (where $F[0,1]$ is the set of real valued functions on the interval $[0,1]$). Is the subset S consisting of functions $f(x)$ such that $f(.5) = 0$ a subspace?

5 YES. (CLOSED UNDER + SINCE $f, g \in F[0,1]$ st. $f(1/2) = g(1/2) = 0$ $(f+g)(1/2) = f(1/2) + g(1/2) = 0 + 0 = 0$.

W/O THIS = 3 CLOSED UNDER S. MULT. SINCE IF $c \in \mathbb{R}$, f AS ABOVE, $cf(1/2) = c \cdot 0 = 0$.

WORTH 1 PT EACH FOR STATING

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4. Which of the following sets are linearly independent and **why/why not**?

4 $\left\{\begin{pmatrix}2\\3\\1\\0\end{pmatrix}, \begin{pmatrix}1\\2\\3\\4\end{pmatrix}\right\}$ 4 $\left\{\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\9\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\3\\7\end{pmatrix}\right\}$ 4 $\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\3\\1\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}\right\}$

INDEPENDENT, NOT SCALAR MULTIPLES OF EACH OTHER

DEPENDENT – MORE VECTORS THAN THERE ARE IN A BASIS

$\det\begin{pmatrix}0&1&1\\1&3&2\\1&1&0\end{pmatrix} = (0+2+1) - (3) = 0$

LIN. DEP. ALSO $\begin{pmatrix}0\\1\\1\end{pmatrix} + \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}1\\3\\1\end{pmatrix}$

4 $\left\{\begin{pmatrix}1&-1\\0&1\end{pmatrix}, \begin{pmatrix}2&1\\1&0\end{pmatrix}, \begin{pmatrix}1&-3\\-1&1\end{pmatrix}\right\}$

$\begin{pmatrix}1&2&1\\-1&1&-3\\0&1&-1\\1&0&1\end{pmatrix} \to \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}$ SO NO FREE VAR. SO LIN. INDEP.

5. Which of the following sets of vectors span their indicated vector spaces and **why/why not**?

a) $p, q, r \in P_3$ with $p = x^3 + x + 1$, $q = x^2 + x + 1$ and $r = 1$

5 NEED 4 VECTORS TO SPAN P_3

ex $1, x, x^2, x^3$ IS A BASIS.

LIN. INDEP INSTEAD OF SPAN = 1

b) $(0, 1, 1), (1, 3, 2)$ and $(1, 1, 0)$ in $\mathbb{R}^3$

5 ~~DOESN'T SPAN~~

$\begin{pmatrix}0&1&1&x\\1&3&1&y\\1&2&0&z\end{pmatrix}$

MATRIX = 2

CAN WE ALWAYS SOLVE?

$\det\begin{pmatrix}0&1&1\\1&3&1\\1&2&0\end{pmatrix} = (0+1+2) - (3+0+0)$

NO, CAN'T ALWAYS SOLVE.

$\left(\begin{array}{ccc|c}0&1&1&x\\1&3&1&y\\1&2&0&z\end{array}\right) \to \left(\begin{array}{ccc|c}1&2&0&z\\1&3&1&y\\0&1&1&x\end{array}\right)$

$\to \left(\begin{array}{ccc|c}1&2&0&z\\0&1&1&y-z\\0&1&1&x\end{array}\right) \to \left(\begin{array}{ccc|c}1&2&0&z\\0&1&1&y-z\\0&0&0&x-y+z\end{array}\right)$

ONLY SOLVABLE IF $x-y+z=0$. IE FOR VECTORS OF FORM $\begin{pmatrix}x\\y\\-x+y\end{pmatrix}$

c) $\begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$, and $\begin{pmatrix}1\\2\end{pmatrix}$ in $\mathbb{R}^2$

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$a_1\begin{pmatrix}1\\3\end{pmatrix} + a_2\begin{pmatrix}0\\1\end{pmatrix} + a_3\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$

$\begin{pmatrix}1&0&1\\3&1&2\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$

$\left(\begin{array}{ccc|c}1&0&1&x\\3&1&2&y\end{array}\right) \to \left(\begin{array}{ccc|c}1&0&1&x\\0&1&-1&y-3x\end{array}\right)$

SO YES, IT DOES SPAN.

6. Let $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ -3 \\ -1 \\ 1 \end{pmatrix}$.

$c_1\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 1 \\ -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ 22

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a) Compute $Span\{v_1, v_2, v_3\}$. Is $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ in the span? 5 YES

$\left(\begin{array}{ccc|c} 1 & 2 & 1 & x \\ -1 & 1 & -3 & y \\ 0 & 1 & -1 & z \\ 1 & 0 & 1 & w \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2x-3y+7z \\ 0 & 1 & 0 & x+y-2z \\ 0 & 0 & 1 & x+y-3z \\ 0 & 0 & 0 & x+2y-4z+w \end{array}\right)$

b) Give a basis for the span you found in part a).

SPAN IS $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ s.t. $x+2y-4z+w=0$ SO $\begin{pmatrix} x \\ y \\ z \\ -x-2y+4z \end{pmatrix}$

$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ IS $x=y=z=w=1$ $-1-2\cdot1+4\cdot1=1$ SO YES, IT IS!

BASIS IS $\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 4 \end{pmatrix}$

~~SPAN OF~~ BASIS FOR $\begin{pmatrix} -2x+3y+7z \\ x+y-2z \\ x+y-3z \\ x+2y-4z+w \end{pmatrix}$ 2pts.

$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 1 & -3 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \end{array}\right) \to \begin{pmatrix} 1 & 0 & 0 \\ & & \\ & & \end{pmatrix}$

$2\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + -1\begin{pmatrix} 1 \\ -3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2-1 \\ -2+3 \\ 0-(-1) \\ 2+(-1) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

7. Let $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & -3 & -1 & 1 \end{pmatrix}$ which has reduced row echelon form $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$.

The dimension of NS(A) is 1 and the rank of A is 3.

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a) Give a non-zero vector in the Nullspace of A.

$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 4 & 0 \end{array}\right)$ SO LET $c_4=1$ THEN

b) Compute the Column Space of A.

CHECK $\begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & -3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ -4 \\ 1 \end{pmatrix} = \begin{pmatrix} 1-2+1 \\ 2+2-4 \\ 1-6+4+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ✓ $\begin{pmatrix} 1 \\ 2 \\ -4 \\ 1 \end{pmatrix}$ IS IN ker.

COLUMN SPACE = Span$\left\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right\}$ =2

$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & x \\ 2 & 1 & 1 & 0 & y \\ 1 & -3 & -1 & 1 & z \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & y-z \\ 0 & 1 & 0 & -2 & y-x-z \\ 0 & 0 & 1 & 4 & 2y-3x-z \end{array}\right)$

SO THEY SPAN $\mathbb{R}^3$!

BASIS FOR ROW SPACE = 2

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8. True/False, circle T or F as appropriate.

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a) T F All examples of vector addition and scalar multiplication on a real vector space are based on ordinary addition and multiplication of real numbers respectively.

$\mathbb{R}^{>0}$

b) T F Our definitions of vector space can be easily be extended to define vector spaces over other sets of scalars such as the rational numbers or the complex numbers.

c) T F Ordinary addition is an example of vector addition.

d) T F If the reduced row echelon form of a matrix whose rows are the vectors $\{v_1, ..., v_2\}$ has a free variable, then the system is linearly dependent.

2 BONUS PTS? CONFUSINGLY WORDED. VECTORS ARE.

e) T F A set of vectors with more vectors than pieces on information in each vector is automatically linearly dependent.

f) T F The determinate is not a useful tool for showing that a set of n vectors in $\mathbb{R}^n$ is linearly independent.

9. More fill in the blank.

8

a) Give two different examples of bases for the vector space P_2.

AS VECTORS = 1 EACH.

$1, x, x^2$ and $x^2+1, x^2+x, 1$.

b) The *dimension* of a vector space V is defined to be SIZE OF ANY BASIS and is well defined because ALL BASES HAVE THE SAME SIZE.

10. Let $f_1(x) = 2x+1$, $f_2(x) = 1$. We can show that the set of function $f_1, f_2 \in F(-\infty, \infty)$ are linearly independent using either the Wronksian or another method. Demonstrate that these two methods give you the same piece of information.

$W(f_1, f_2) = \det\begin{vmatrix} 2x+1 & 1 \\ 2 & 0 \end{vmatrix} = 2 \neq 0$ SO FUNCTIONS ARE LIN. INDEP.

MAYBE JUST 5?

$2x+1$ & 1 ARE VECTORS IN P_2.

5

CAN WRITE THEM AS $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ AND $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ WHICH ARE LINEARLY INDEP. VECTORS CAUSE THEY AREN'T S. MULTIPLES OR ECHELON OR $\left(\begin{array}{cc|c} 2 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$ etc.

Reduced Row Echelon Form

The matrix $\left(\begin{array}{rrr|c} 1 & 2 & 1 & x \\ -1 & 1 & -3 & y \\ 0 & 1 & -1 & z \\ 1 & 0 & 1 & w \end{array}\right)$ has reduced row echelon form $\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2x-3y+7z \\ 0 & 1 & 0 & x+y-2z \\ 0 & 0 & 1 & x+y-3z \\ 0 & 0 & 0 & x+2y-4z+w \end{array}\right)$.

The matrix $\left(\begin{array}{rrrr|c} 1 & -1 & 0 & 1 & x \\ 2 & 1 & 1 & 0 & y \\ 1 & -3 & -1 & 1 & z \end{array}\right)$ has reduced row echelon form $\left(\begin{array}{rrrr|c} 1 & 0 & 0 & -1 & y-z \\ 0 & 1 & 0 & -2 & y-x-z \\ 0 & 1 & 0 & 4 & 2y-3x-z \end{array}\right)$