238 Chapter 4 Test

59

March 26, 2013 **Name:** ______________________

By printing my name I pledge to uphold the honor code.

17

Unless otherwise specified, *solution* refers to a real valued function.

1. Fill in the blank.

2 a) All solutions to a linear ODE are of the form $y_p + y_H$ where y_p is a solution to ORIG ODE and y_H is a solution to HOMOGENEOUS ODE.

2 b) The solutions to an n th order LINEAR HOMOGENEOUS ODE form an n-dimensional vector space.

3 c) Our technique for solving homogeneous linear ODE's only applies to ones with CONSTANT COEFFICIENTS and relies on finding the ROOTS of the CHARACTERISTIC POLYNOMIAL.

5 **2.** Determine the general solution for the following differential equations.

a) $y''' - y'' + y' - y = 0$

1 THIS FAR = 2

CHAR POLY IS $x^3 - x^2 + x - 1 = (x-1)(x^2+1)$

WHICH HAS $x = 1$ FOR A ROOT

HAS ROOTS $\lambda = 1, \lambda = i, \lambda = -i$

$$\begin{array}{r} x^2 + 1 \\ x-1 \,\overline{)\, x^3 - x^2 + x - 1} \\ -x^3 - x^2 \\ \hline x - 1 \\ -x - 1 \\ \hline \end{array}$$

SO e^{1x}, e^{ix}, e^{-ix} ARE SOLUTIONS. e^{ix} AREN'T REAL VALUED, BUT $e^{ix} = \cos x + i\sin x$ SO

1/2 PT EACH? 3 1/2

1 1/2

$\boxed{y = a_1 e^{x} + a_2 \cos x + a_3 \sin x}$ $a_1, a_2, a_3 \in \mathbb{R}$

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5 b) $y'' - 4y' + 4y = e^{-x}$

SOLVE $y'' - 4y' + 4y = 0$ 1

CHAR POLY $x^2 - 4x + 4 = (x-2)^2$

SO SOLUTIONS ARE e^{2x} & xe^{2x}

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METHOD OF UNDETERMINED COEFFS.

$y_p = Ae^{-x}$

$y_p' = -Ae^{-x}$

$y_p'' = Ae^{-x}$

$y'' - 4y' + 4y = Ae^{-x} + 4Ae^{-x} + 4Ae^{-x} = e^{-x}$

$9Ae^{-x} = e^{-x}$ $A = 1/9$

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GENERAL SOLUTION

$y = \frac{1}{9}e^{-x} + c_1 e^{2x} + c_2 x e^{2x}$!!!

$y_p = A\cos x + B\sin x$

$y_p' = -A\sin x + B\cos x$

$y_p'' = -A\cos x - B\sin x$

$y'' - 2y' + 2y = -A\cos x - B\sin x - 2(-A\sin x + B\cos x) + 2(A\cos x + B\sin x)$

$= (-A - 2B + 2A)\cos x + (-B + 2A + 2B)\sin x$

$= (A - 2B)\cos x + (B + 2A)\sin x$

$A - 2B = 1$

$B + 2A = 0 \quad B = -2A \qquad A - 2(-2A) = 1$

$4A + 2B = 0$

$5A = 1$

$A = \frac{1}{5} \quad B = -\frac{2}{5}$

$y = c_1 e^x \cos x + c_2 e^x \sin x + \frac{1}{5}\cos x - \frac{2}{5}\sin x$

$y(0) = c_1 e^0 \cos 0 + \cancel{c_2 e^0 \sin 0} + \frac{1}{5}\cos 0 - \cancel{\frac{2}{5}\sin 0}$

$= c_1 + \frac{1}{5} = 1 \quad c_1 = \frac{4}{5}$

$y = \frac{4}{5}e^x \cos x + c_2 e^x \sin x + \frac{1}{5}\cos x - \frac{2}{5}\sin x$

$y' = \frac{4}{5}e^x \cos x - \frac{4}{5}e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x - \frac{1}{5}\sin x - \frac{2}{5}\cos x$

$y'(0) = \frac{4}{5} + c_2 - \frac{2}{5} = \frac{2}{5} + c_2 = \frac{10}{5}$

$c_2 = \frac{8}{5}$

$y = \frac{4}{5}e^x \cos x + \frac{8}{5}e^x \sin x + \frac{1}{5}\cos x - \frac{2}{5}\sin x$

$w_1 = \begin{vmatrix} 0 & x \\ 1/x^3 & 1 \end{vmatrix} = -1/x^2 \qquad w = \frac{2}{x}$

$w_2 = \begin{vmatrix} \frac{1}{x} & 0 \\ \frac{-1}{x^2} & 1/x^3 \end{vmatrix} = \frac{1}{x^4}$

$u_1 = \int \frac{-1/x^2}{\frac{2}{x}}\,dx = \int -\frac{1}{2}\frac{1}{x}\,dx = -\frac{1}{2}\ln|x|$

$u_2 = \int \frac{\frac{1}{x^4}}{\frac{2}{x}}\,dx = \frac{1}{2}\int \frac{1}{x^3}\,dx = -\frac{1}{4}x^{-2} = \frac{1}{4x^2}$

ORDER	DO	IVP	LAST 3
IVP		2	
y_H	3	$k=2$	
y_P	3		

3. Solve the following initial value problem. $y = -\frac{1}{3}\cos x - \cancel{\frac{2}{3}\sin x + \frac{4}{3}e^x\cos x} + \frac{4}{3}e^x \sin x$

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$$y'' - 2y' + 2y = \cos x,\ y(0) = 1,\ y'(0) = 2$$

CHAR POLY x^2-2x+2 DOESN'T FACTOR — HAS ROOTS

$x = \frac{2 \pm \sqrt{4-4\cdot 2}}{2} = \frac{2\pm\sqrt{-4}}{2} = 1 \pm i$ SO SOLUTIONS

$e^{(1+i)x}, e^{(1-i)x}$ OR $e^{(1+i)x} = e^x \cdot e^{ix} = e^x(\cos x + i \sin x)$

SOLUTIONS ARE $y_1 = e^x \cos x$ $y_2 = e^x \sin x$.

PARTICULAR SOLUTION

$y_p = A\cos x + B \sin x$ $y'' - 2y' + 2y = -A\cos x - B\sin x - 2(B\cos x - A\sin x) + 2(A\cos x + B\sin x)$

$y_p' = B\cos x - A\sin x$ $= (-A - 2B + 2A)\cos x + (-B + 2A + 2B)\sin x = \cos x.$

$y_p'' = -A\cos x - B\sin x$

$A - 2B = 1$

$-2A + B = 0$

$-3A = 1$

$A = -1/3$

$B = -2/3$

$y = -\frac{1}{3}\cos x - \frac{2}{3}\sin x + c_1 e^x\cos x + c_2 e^x \sin x$

$y(0) = -\frac{1}{3} + c_1 = 1$ $c_1 = 4/3$.

$y' = -\frac{2}{3}\cos x + \frac{1}{3}\sin x + \frac{4}{3}e^x\cos x - \frac{4}{3}e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x$

$y'(0) = -\frac{2}{3} + \frac{4}{3} + c_2$ $c_2 = 4/3$

4. Non constant coefficient ODEs.

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a) Show that the given set of functions forms a fundamental solution set to the homogeneous ODE.

$$\left\{\frac{1}{x}, x\right\}, x^2y'' + xy' - y = 0$$

CHECK THEY ARE SOLUTIONS

$y = \frac{1}{x} = x^{-1}$

$y' = \frac{-1}{x^2} = -x^{-2}$

$y'' = +\frac{2}{x^3} = 2x^{-3}$

$x^2\left(\frac{2}{x^3}\right) + x\left(\frac{-1}{x^2}\right) - \frac{1}{x} = 0$

$y = x$

$y' = 1$

$y'' = 0$

$x^2 \cdot 0 + x \cdot 1 - x = 0$

CHECK THEY ARE LIN. INDEP.

LIN. INDEP.

$W(\frac{1}{x}, x) = \left|\begin{pmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{pmatrix}\right| = \frac{1}{x} - \left(\frac{-x}{x^2}\right) = \frac{2}{x} \neq 0$ ALWAYS

SO THEY ARE LIN. INDEP.

2 $\frac{c_1}{x} + c_2 x = 0 \Rightarrow$ 3 $\frac{c_1 + xc_2}{x} = 0$ $c_1 + xc_2 = 0$ $\forall x$. ONLY POSSIBLE IF $c_1 = c_2 = 0$

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b) Use the information from part a to find a particular solution to the following non-homogeneous ODE.

$$x^2y'' + xy' - y = \frac{1}{x}$$

$f_1 = -\frac{1}{2}\ln|x|$

$f_2 = \frac{-1}{4x^2}$

$\boxed{\frac{-\frac{1}{2}\ln|x|}{x} + \frac{-1}{4x}}$

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$y_p = f_1 \cdot \frac{1}{x} + f_2 \cdot x$

$y_p' = \cancel{f_1' \cdot \frac{1}{x}} + f_1 \cdot \frac{-1}{x^2} + \cancel{f_2' x} + f_2$

$y_p'' = f_1' \frac{-1}{x^2} + f_1 \cdot \frac{2}{x^3} + f_2'$

TOGETHER GO TO 0

ASSUME $f_1' \frac{1}{x} + f_2' x = 0$

SO $x^2\left(\frac{-f_1'}{x^2} + f_2'\right) + 0 - 0 = \frac{1}{x}$. SOLVE.

$\begin{pmatrix} \frac{1}{x} & x \\ \frac{-1}{x^2} & 1 \end{pmatrix}\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{x^3} \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1/x \end{pmatrix} = -1$

$\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = \frac{x}{2}\begin{pmatrix} 1 & -x \\ 1/x^2 & \frac{1}{x} \end{pmatrix}\begin{pmatrix} \frac{1}{x} & x \\ -1/x^2 & 1 \end{pmatrix}\begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = \frac{x}{2}\begin{pmatrix} 1 & -x \\ \frac{1}{x^2} & \frac{1}{x} \end{pmatrix}\begin{pmatrix} 0 \\ \frac{1}{x^3} \end{pmatrix} = \left\{\frac{x}{2}\begin{pmatrix} \frac{-1}{x^2} \\ \frac{1}{x^4} \end{pmatrix}\right\} = \begin{pmatrix} -\frac{1}{2x} \\ \frac{1}{2x^3} \end{pmatrix}$

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5. True/False, circle T or F as appropriate.

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a) **T** **F** Phase portraits are not useful for n^{th} order ODEs because the phase portrait would have to live in $\mathbb{R}^n$ because of all the data necessary for an IVP.

b) **T** **F** The sum of two solutions to a general homogeneous ODE is still a solution to that equation. ONLY TRUE FOR LINEAR!

c) **T** **F** If λ is a root of the characteristic polynomial for linear constant coefficient homogeneous ODE, then $e^{\lambda x}$ is always a solution to the (homogeneous) ODE, even if it is not a real valued function.

d) **T** **F** If a complex valued function is a solution to a homogeneous linear ODE, then both is real part and its imaginary part are individually solutions.

e) **T** **F** The above is false for non-linear homogeneous ODEs.

6. More fill in the blanks.

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a) An initial value problem for an nth order ODE involves giving n extra pieces of information, usually in the form of $y(x_0)=k_0, y'(x_0)=k_1, \ldots, y^{(n-1)}(x_0)=k_{n-1}$.

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b) The characteristic polynomial for a constant coefficient homogeneous linear ODE is extracted by substituting $y =$ $e^{\lambda x}$ into the ODE.

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c) Given the non-homogeneous linear differential equation $y'' + 3y' + 5y = e^x \cos x$ the method of undetermined coefficients would give our first guess for

$y_P =$ $Ae^x\cos x + Be^x\sin x$.

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d) The non-homogeneous linear ODE $y'' + 3y' + 5y = \sec x$ is not a good choice for the method

$g(x)=\sec x,\ g'(x)=\sec x\tan x,\ g''(x)=\sec^2 x\tan x + \sec^3 x$

of undetermined coefficients because DERIVATIVES OF SEC X DON'T. FALL INTO etc. ONE OF A HANDFUL OF TYPES

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e) The method of variation of parameters involves setting

$y_P =$ $f_1y_1 + f_2y_2 + \cdots + f_ny_n$ where $y_1, y_2, \ldots, y_n$ SOLUTIONS TO HOMOGENEOUS SYSTEM. and

solving for $f_1, \ldots, f_n$ by making the simplifying assumption that

$f_1'y_1 + f_2'y_2 + \cdots + f_n'y_n = 0$

$f_1'y_1' + f_2'y_2' + \cdots + f_n'y_n' = 0$ etc.

(OK TO JUST HAVE deg 2)

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? 4, 6 ?