238 Chapter 5 Test

April 16, 2013

Name: KEY

By printing my name I pledge to uphold the honor code.

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1. Fill in the blank.

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a) A function $T : V \to W$ between vector spaces V and W is a linear transformation if the following two equations are satisfied: $T(u+v)=T(u)+T(v)$ holds for all $u, v \in V$ and $T(c \cdot u)=c \cdot T(u)$ holds for all $\forall u \in V,\ c \in \mathbb{R}$.

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b) A linear transformation $T : V \to W$ is completely determined by its value on ANY BASIS FOR V.

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c) Eigenvalues for a matrix A are found by computing ROOTS OF $\det(\lambda I - A)$ while eigenspaces are found by computing $NS(\lambda I - A)$.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y + z \\ 3y \end{pmatrix}$.

a) Prove that T is a linear transformation using the definition.

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$$T\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{pmatrix} =$$

$$\text{vs. } T\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} + T\begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_1-y_1+z_1 \\ 3y_1 \end{pmatrix} + \begin{pmatrix} x_2-y_2+z_2 \\ 3y_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1+x_2-y_1-y_2+z_1+z_2 \\ 3(y_1+y_2) \end{pmatrix}$$

$$T\left(c\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = T\begin{pmatrix} cx \\ cy \\ cz \end{pmatrix} = \begin{pmatrix} cx-cy+cz \\ 3cy \end{pmatrix}$$

$$= c\begin{pmatrix} x-y+z \\ 3y \end{pmatrix}$$

$$= c\,T\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

b) Let α be the standard basis for $\mathbb{R}^3$ and β be the standard basis for $\mathbb{R}^2$. Find $[T]_\alpha^\beta$.

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$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y+z \\ 3y \end{pmatrix}$$

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3. Given $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ and $T\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

a) Given the matrix for T in terms of the standard basis.

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\quad T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{2}{3}T\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{3}T\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 0 \\ 5 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$a - b = 1$ $\quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix} + d\begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$a + 2b = 0$ $\quad c - d = 0$

$3a = 2$ $\quad a = \frac{2}{3}$ $\quad b = \frac{-1}{3}$ $\quad c + 2d = 1$ $\quad 3c = 1$ $\quad c = \frac{1}{3}$ $\quad d = \frac{1}{3}$

$T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 \\ 5 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

$$\boxed{[T] = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}}$$

$P = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$, $P^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$

OR

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ P \uparrow \downarrow P^{-1} & \nearrow & \\ \mathbb{R}^2_{\beta} & \begin{pmatrix} 0 & -3 \\ 5 & 1 \end{pmatrix} = [T]_\alpha & \end{array}$$

$\begin{pmatrix} 0 & -3 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}$

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b) Give a matrix $[T]^\beta_\beta$ such that $[T]^\beta_\beta$ is in Jordan canonical form for some basis β.

ie FIND ITS EIGENVALUES

$\det\left(\lambda I - \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix}\right) = \det\begin{pmatrix} \lambda-1 & 1 \\ -3 & \lambda-2 \end{pmatrix}$

$= (\lambda-1)(\lambda-2)+3$

$= \lambda^2 - 3\lambda + 5$

$\lambda = \frac{3 \pm \sqrt{9-4\cdot 5}}{2} = \frac{3 \pm \sqrt{-11}}{2} = \frac{3}{2} \pm \frac{\sqrt{11}}{2}i$

$$[T]^\beta_\beta = \begin{pmatrix} \frac{3}{2} + \frac{\sqrt{11}}{2}i & 0 \\ 0 & \frac{3}{2} - \frac{\sqrt{11}}{2}i \end{pmatrix}$$

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4. The characteristic polynomial of the matrix $A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$ is $(\lambda+1)^2(\lambda-8)$.

$A - \lambda I.$

a) Compute all of the eigenspaces for the matrix A.

$E_{-1} = NS\begin{pmatrix} -4 & -2 & -4 \\ -2 & -1 & -2 \\ -4 & -2 & -4 \end{pmatrix}$

$= NS\begin{pmatrix} 1 & 1/2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left\{\begin{pmatrix} -\frac{1}{2}c_2 - c_3 \\ c_2 \\ c_3 \end{pmatrix}\right\}$

$c_1 + \frac{1}{2}c_2 + c_3 = 0$

c_2 & c_3 FREE VAR

$$E_{-1} = \left\{\begin{pmatrix} -\frac{1}{2}c_2 - c_3 \\ c_2 \\ c_3 \end{pmatrix} \middle| c_2, c_3 \in \mathbb{R}\right\}$$

$$E_{8} = \left\{\begin{pmatrix} c_3 \\ 1/2\, c_3 \\ c_3 \end{pmatrix} \middle| c_3 \in \mathbb{R}\right\}$$

$$E\_\_ =$$

$\begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix}$

$E_8 = NS\begin{pmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{pmatrix}$

$= NS\begin{pmatrix} \cancel{5 \;\; -2 \;\; -4} \\ 1 & -4 & 1 \\ 0 & -18 & -9 \\ 0 & -18 & 9 \end{pmatrix}$

$= NS\begin{pmatrix} 1 & -4 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix} = NS\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$

$c_1 - c_3 = 0$

$c_2 - 1/2\, c_3 = 0$

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b) Find P and D such that D is diagonal and $A = PDP^{-1}$.

$$\begin{array}{ccc} \mathbb{R}^3 & \xrightarrow{\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}} & \mathbb{R}^3 \\ \downarrow & & \downarrow \\ \mathbb{R}^3 & \xrightarrow{\begin{pmatrix} -1 & & \\ & -1 & \\ & & 8 \end{pmatrix}} & \mathbb{R}^3 \end{array}$$

$\beta = \left\{\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\right\}$

$$D = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 8 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & -1 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

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5. True/False, circle T or F as appropriate. *If you have time after you have finished the rest of the exam* you may go back and explain your answers for possible partial credit/extra credit. 20

a) **T** **F** If $T(\mathbf{0}) \neq \mathbf{0}$, then T is not a linear transformation.

b) **T** **F** The definition of matrix multiplication from Chapter 1 comes from what we would need for composition of linear transformations to be multiplication of the cooresponding matrices.

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c) **T** **F** Every matrix has a unique Jordan canonical form (up to reordering the jordan blocks).

d) **T** **F** A two by two matrix with repeated eigenvalues is either already diagional or not diagionalizable.

e) **T** **F** $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector for $\begin{pmatrix} 1 & -1 & 2 & 1 \\ -6 & 9 & -1 & 1 \\ 1 & 2 & -3 & 3 \\ 4 & -2 & 1 & 0 \end{pmatrix}$. $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix}$

6. More fill in the blanks.

a) A matrix is diagionalizable if and only if

LOTS OF OPTIONS.

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b) The dimension of the eigenspace $E_\lambda \leq$ MULTIPLICITY OF EIGENVALUES.

c) Describe one way the diagional form of a matrix can be useful.

RAISING MATRICES TO LARGE POWERS.

7. Describe the linear transformation from problem 3 in terms of what the transformation does to $\mathbb{R}^2$.

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